COSC 2011 3.0 Fundamentals of Data Structures

Section N, Winter 2001

Class Notes: **Mathematical Induction**

Solutions to Practice Questions

1) Use mathematical induction to prove the inequality $n < 2^n$ for all positive integers n.

Solution: Let $P(n)$ denote the statement "$n < 2^n$". We must first complete the basis step; that is, we must show $P(1)$ is true. Then we must carry out the inductive step; that is, we must show $P(n+1)$ is true when $P(n)$ is true.

BASIS STEP: $P(1)$ is true since $1 < 2^n = 2$

INDUCTIVE STEP: Assume that $P(n)$ is true for the positive integer n. That is, assume $n < 2^n$. We need to show $P(n+1)$ is true. That is, we need to show $n+1 < 2^{n+1}$. Adding 1 to both sides of $n < 2^n$, and then noting that $1 \leq 2^n$, gives:

$$n + 1 < 2^n + 1 \leq 2^n + 2^n = 2^{n+1}$$

This shows that P(n+1) is true, namely, that $n + 1 < 2^{n+1}$, based on the assumption that $P(n)$ is true. The induction step is complete. Therefore by the principle of mathematical induction, it has been shown that $n < 2^n$ is true for all positive integers n.

2) Use mathematical induction to prove that $2^n < n!$ for every positive integer n, with $n \geq 4$.

Solution: Let $P(n)$ denote the statement "$2^n < n!$". We must first complete the basis step; that is, we must show $P(1)$ is true. Then we must carry out the inductive step; that is, we must show $P(n+1)$ is true when $P(n)$ is true.

BASIS STEP: To prove the inequality for $n \geq 4$ requires that the basis step be $P(4)$. Note that $P(4)$ is true since $2^4 = 16 < 4! = 24$.

INDUCTIVE STEP: Assume that $P(n)$ is true for the positive integer n. That is, assume $2^n < n!$. We need to show $P(n+1)$ is true. That is, we need to show $2^{n+1} < (n+1)!$. Multiplying both sides of the inequality $2^n < n!$ by 2, it follows that:

$$2 \times 2^n < 2 \times n! < (n+1) \times n! = (n+1)!$$

This shows that P(n+1) is true, namely, that $2^n < n!$, based on the assumption that $P(n)$ is true. The induction step is complete. Therefore by the principle of mathematical induction, it has been shown that $2^n < n!$ is true for all positive integers n.