COSC 2011 3.0 Fundamentals of Data Structures

Section N, Winter 2001

Class Notes: **Mathematical Induction**

An Example of a "Proper" Proof Using Mathematical Induction

Use mathematical induction to prove the sum of the first n odd positive integers is n^2

Solution: Let $P(n)$ denote the statement that the sum of the first n odd positive integers is n^2. We must first complete the basis step; that is, we must show $P(1)$ is true. Then we must carry out the inductive step; that is, we must show $P(n+1)$ is true when $P(n)$ is true.

BASIS STEP: $P(1)$ states that the sum of the first one odd integers is $1^2 = 1$. This is obviously true since the sum of the first odd positive integer is 1.

INDUCTIVE STEP: To complete the inductive step we must show that the proposition $P(n) \to P(n+1)$ is true for every positive integer n. To do this, suppose (assume) that $P(n)$ is true for a positive integer n; that is,

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

(Note that the nth odd positive integer is (2n-1), since the integer is obtained by adding 2 a total of n-1 times to 1). We must show that $P(n+1)$ is true, assuming that $P(n)$ is true. $P(n+1)$ is the statement that

$$1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1) = (n + 1)^2$$

So, assuming that $P(n)$ is true, it follows that

$$\begin{aligned} 1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1) &= [1 + 3 + \ldots + (2n - 1)] + (2n + 1) \\ &= n^2 + (2n + 1) \\ &= n^2 + 2n + 1 \\ &= (n + 1)^2 \end{aligned}$$

This shows that P(n+1) follows from P(n). Note that we used the inductive hypothesis P(n) in the second equality to replace the sum of the first n odd positive integers by n^2.

Since $P(1)$ is true and the implication $P(n) \to P(n+1)$ is true for all positive integers n, the principle of mathematical induction shows that $P(n)$ is true for all positive integers n.

Another Example: (Running Time of Bubble-Sort)

Justify the following equality using induction.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

BASIS STEP: $n = 1$. Trivial, for $1 = \frac{n(n+1)}{2}$, if $n = 1$.

INDUCTIVE STEP: $n \geq 2$. Assume the claim is true for $n' < n$. Consider n.

$$\sum_{i=1}^{n} i = n + \sum_{i=1}^{n-1} i$$

By the inductive hypothesis, then

$$\sum_{i=1}^{n} i = n + \frac{(n-1)n}{2}$$

which we can simplify as

$$n + \frac{(n-1)n}{2} = \frac{2n + n^2 - n}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}$$

This completes the justification.

Practice Questions:

1) Use mathematical induction to prove the inequality $n < 2^n$ for all positive integers n.

2) Use mathematical induction to prove that $2^n < n!$ for every positive integer n, with $n \geq 4$.