COSC 2011 Section N

Tuesday, May 8 2001

Overview

- Undirected Graph Traversals
  - Depth-First Search
  - Breadth-First Search

Undirected Graph Traversal - DFS:

- Definition:
  - A ***graph traversal*** is a systematic procedure for visiting all vertices and edges of a graph.
  - Efficient if it visits all vertices and edges in linear time.
  - Two efficient methods:
    - Depth-First Search
    - Breadth-First Search

Undirected Graph Traversal - DFS:

- Depth-First Search (DFS):
  - “Search” deeper in the graph whenever possible.
  - Edges are “explored” out of the the most recently visited vertex *v* that still has unexplored edges leaving it.
  - When all of *v*’s edges have been explored, search “backtracks” to

Undirected Graph Traversal - DFS:

- explore edges leaving the vertex from which *v* was discovered.
- This process continues until all vertices reachable from the original source vertex have been discovered.
- If any undiscovered vertices remain, one of them is selected as a new source and search repeats.

Exploring a Labyrinth Without Getting Lost

- A **depth-first search (DFS)** in an undirected graph G is like wandering in a labyrinth with a string and a can of red paint without getting lost.
- We start at vertex s, tying the end of our string to the point and painting s "visited". Next we label s as our current vertex called u.
- Now we travel along an arbitrary edge (u,v).
- If edge (u,v) leads us to an already visited vertex v we return to u.
- If vertex v is unvisited, we unroll our string and move to v, paint v "visited", set v as our current vertex, and repeat the previous steps.
- Eventually, we will get to a point where all incident edges on u lead to visited vertices. We then backtrack by unrolling our string to a previously visited vertex v. Then v becomes our current vertex and we repeat the previous steps.

Exploring a Labyrinth Without Getting Lost (cont.)

- Then, if we all incident edges on v lead to visited vertices, we backtrack as we did before. We continue to backtrack along the path we have traveled, finding and exploring unexplored edges, and repeating the procedure.
- When we backtrack to vertex s and there are no more unexplored edges incident on s, we have finished our DFS search.

Undirected Graph Traversal - DFS:

- Visualize DFS by orienting edges along the directions they are explored during the traversal.
  - ***Discovery*** or ***tree*** Edges:
    - Edges used to discover new vertices.
  - ***Back*** Edges:
    - Edges leading to already visited vertices.

Undirected Graph Traversal - DFS:

- Discovery edges form a ***spanning tree*** of the connected component starting at start vertex s.

Undirected Graph Traversal - DFS:

- Algorithm:

Algorithm DFS(v);
Input: A vertex v in a graph
Output: A labeling of the edges as "discovery" edges and "backedges"
for each edge e incident on v **do**
  if edge e is unexplored **then**
    let w be the other endpoint of e
    if vertex w is unexplored **then**
      label e as a discovery edge
      recursively call **DFS**(w)
    else
      label e as a backedge

2001-05-08 COSC 2011 Section 9

Undirected Graph Traversal - DFS:

- Algorithm Assumptions:
  - Have a "way" to determine whether a vertex or edge has been explored or not.
  - Have a "way" to label edges as discovery or back edges.
  - This may require additional storage space and may affect running time!

2001-05-08 COSC 2011 Section 10

DFS Properties

- Proposition 9.12 : Let G be an undirected graph on which a DFS traversal starting at a vertex s has been preformed. Then:
  1) The traversal visits all vertices in the connected component of s
  2) The discovery edges form a spanning tree of the connected component of s
- Justification of 1):
  - Let's use a contradiction argument: suppose there is at least on vertex v not visited and let w be the first unvisited vertex on some path from s to v.
  - Because w was the first unvisited vertex on the path, there is a neighbor u that has been visited.
  - But when we visited u we must have looked at edge(u, w). Therefore w must have been visited.
  - and justification
- Justification of 2):
  - We only mark edges from when we go to unvisited vertices. So we never form a cycle of discovery edges, i.e. discovery edges form a tree.
  - This is a spanning tree because DFS visits each vertex in the connected component of s

Depth-First Search 16

2001-05-08 COSC 2011 Section 11

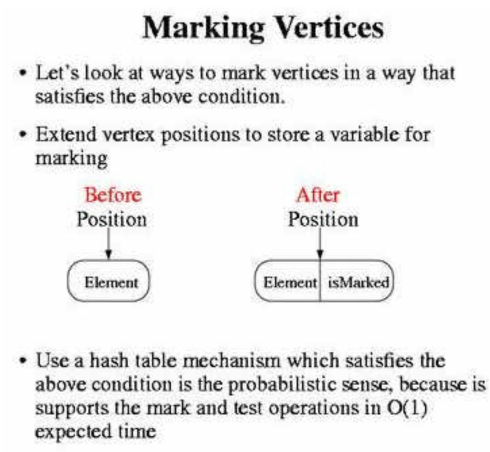
Undirected Graph Traversal - DFS:

- Running Time:

- Remember:
  - DFS is called on each vertex exactly once.
  - Every edge is examined exactly twice, once from each of its vertices
- For n_s vertices and m_s edges in the connected component of the vertex s, a DFS starting at s runs in $O(n_s + m_s)$ time if:
  - The graph is represented in a data structure, like the adjacency list, where vertex and edge methods take constant time
  - Marking a vertex as explored and testing to see if a vertex has been explored takes O(degree)
  - By marking visited nodes, we can systematically consider the edges incident on the current vertex so we do not examine the same edge more than once.

2001-05-08 COSC 2011 Section 12

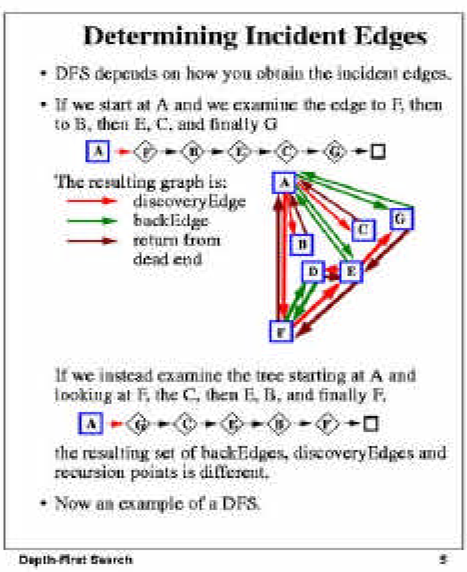
Undirected Graph Traversal - DFS:



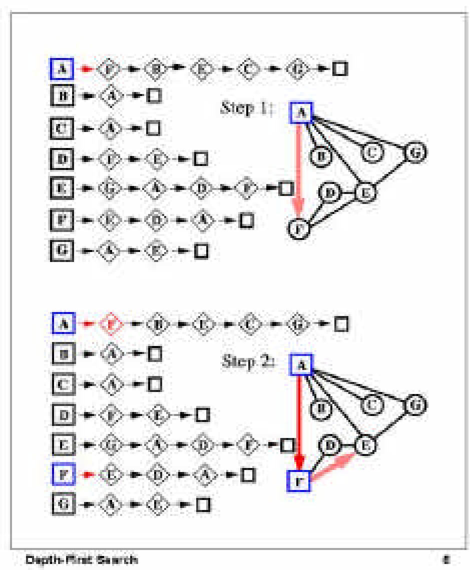
Undirected Graph Traversal - DFS:

- Let G be a graph with n vertices and m edges represented with an adjacency list structure. There exists O(n+m) algorithms based on DFS to compute:
  - Test whether G is connected.
  - Compute ***spanning tree*** of G if G is connected.
  - Compute connected components of G.
  - Compute path between two vertices of G or report no path such path exists.
  - Compute cycle in G or report no cycle exists.

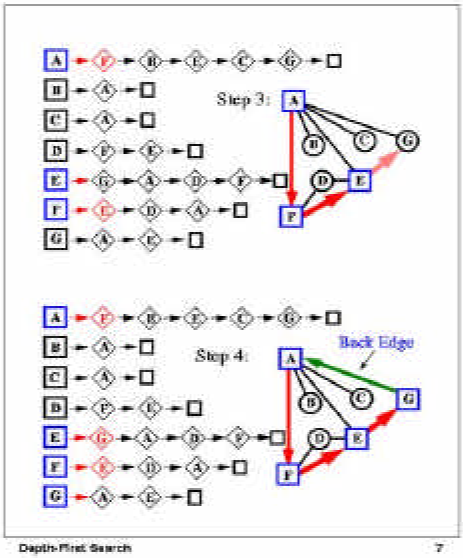
DFS Example: (1)



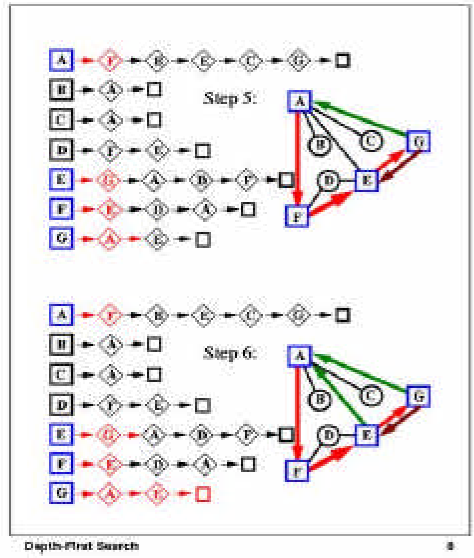
DFS Example: (2)



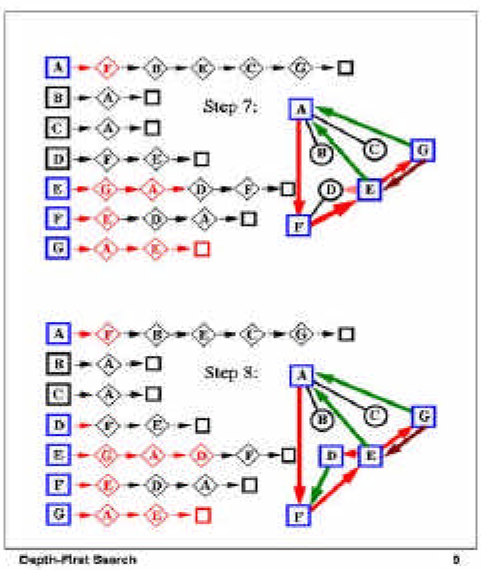
DFS Example: (3)



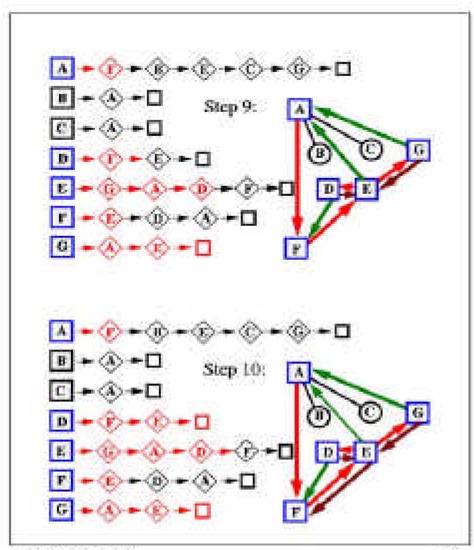
DFS Example: (4)



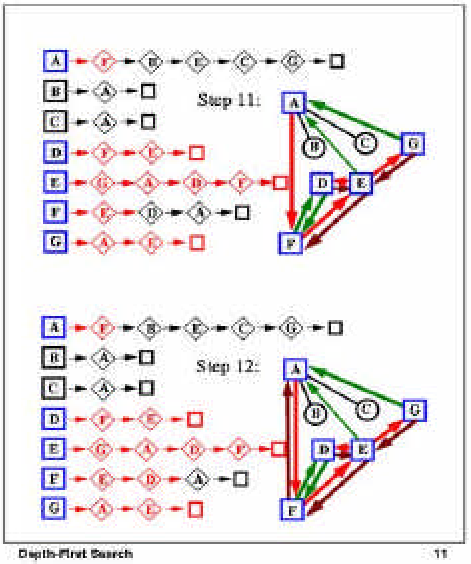
DFS Example: (5)



DFS Example: (6)

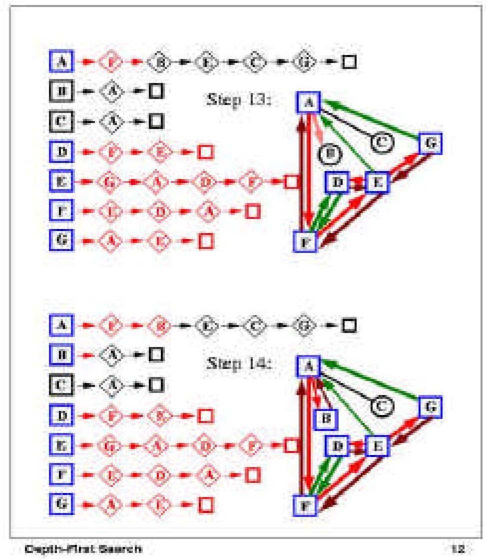


DFS Example: (7)



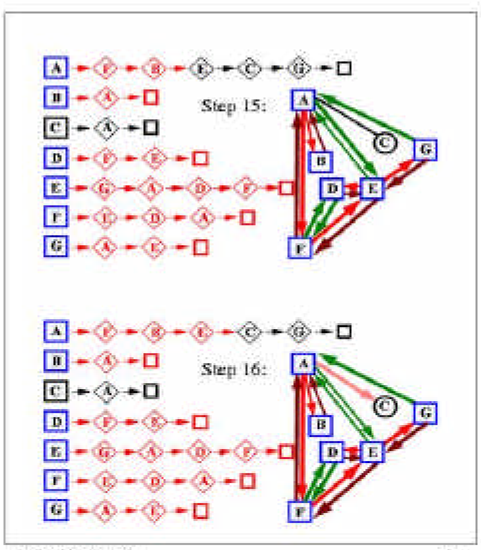
2001-05-08 COSC 2011 Section 21

DFS Example: (8)



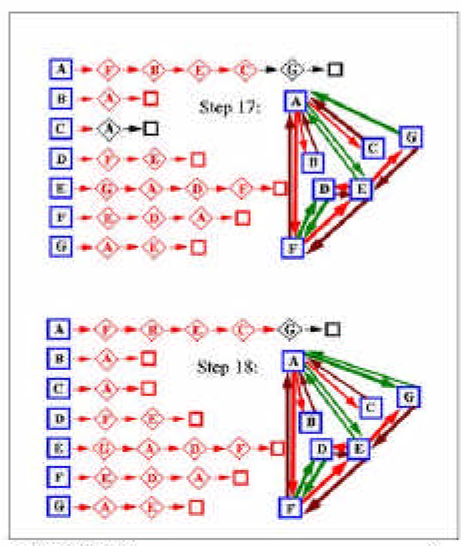
2001-05-08 COSC 2011 Section 22

DFS Example: (9)



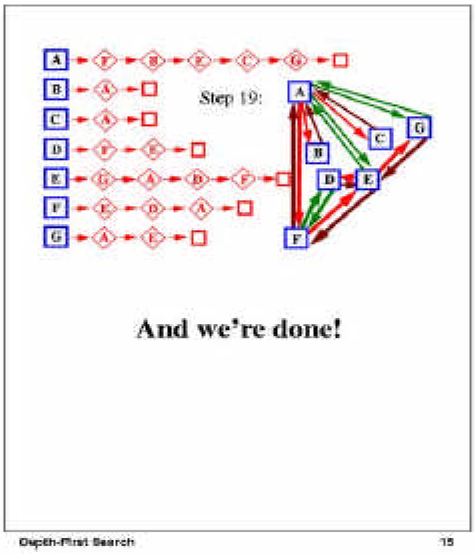
2001-05-08 COSC 2011 Section 23

DFS Example: (10)



2001-05-08 COSC 2011 Section 24

DFS Example: (11)



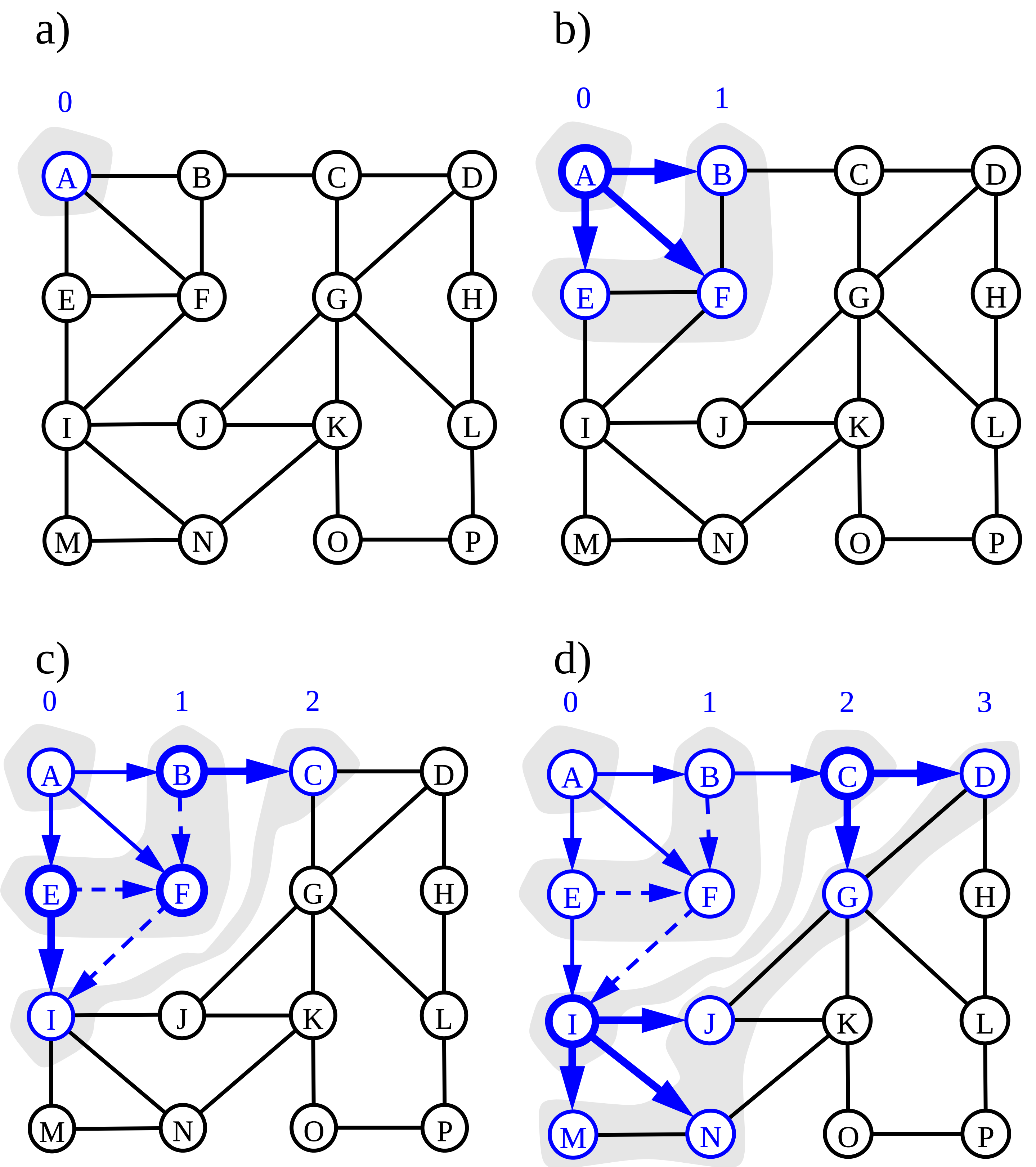
2001-05-08 COSC 2011 Section 25

Breadth-First Search

- Like DFS, a Breadth-First Search (BFS) traverses a connected component of a graph, and in doing so defines a spanning tree with several useful properties

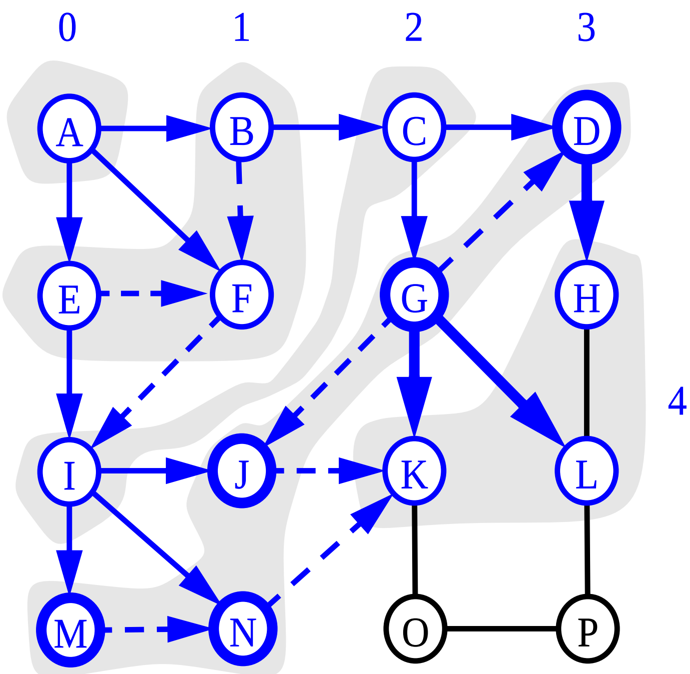
  - The starting vertex s has level 0, and, as in DFS, defines that point as an "anchor."
  - In the first round, the string is unrolled the length of one edge, and all of the edges that are only one edge away from the anchor are visited.
  - These edges are placed into level 1
  - In the second round, all the new edges that can be reached by unrolling the string 2 edges are visited and placed in level 2.
  - This continues until every vertex has been assigned a level.
  - The label of any vertex v corresponds to the length of the shortest path from s to v.

BFS - A Graphical Representation

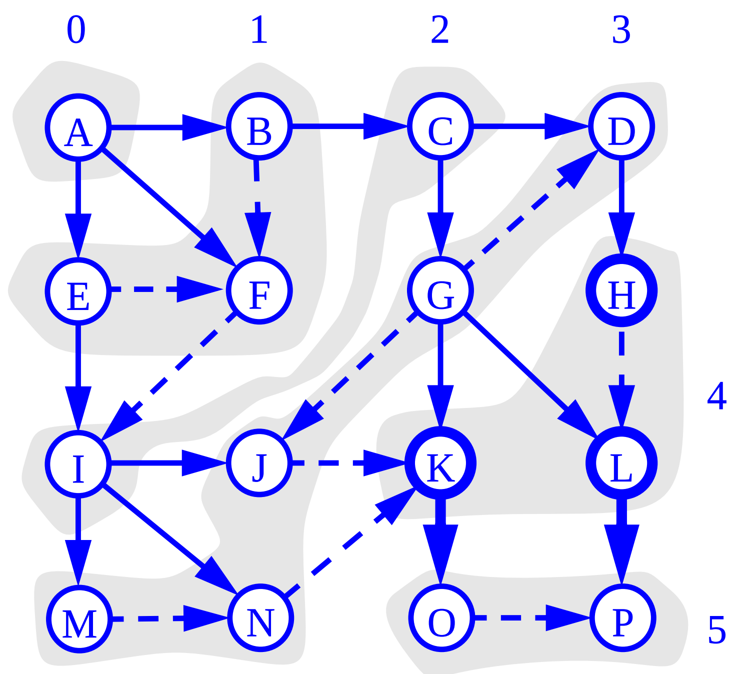


More BFS

e)



f)



BFS Pseudo-Code

```
Algorithm BFS(s):
  Input: A vertex s in a graph
  Output: A labeling of the edges as "discovery" edges
    and "cross edges"
  initialize container L0 to contain vertex s
  i ← 0
  while Li is not empty do
    create container Li+1 to initially be empty
    for each vertex v in Li do
      for each edge e incident on v do
        if edge e is unexplored then
          let w be the other endpoint of e
          if vertex w is unexplored then
            label e as a discovery edge
            insert w into Li+1
          else
            label e as a cross edge
    i ← i + 1
```

Properties of BFS

- Proposition: Let G be an undirected graph on which a a BFS traversal starting at vertex s has been performed. Then
  - The traversal visits all vertices in the connected component of s.
  - The discovery-edges form a spanning tree T, which we call the BFS tree, of the connected component of s
  - For each vertex v at level i, the path of the BFS tree T between s and v has i edges, and any other path of G between s and v has at least i edges.
  - If (u, v) is an edge that is not in the BFS tree, then the level numbers of u and v differ by at most one.
- Proposition: Let G be a graph with n vertices and m edges. A BFS traversal of G takes time $O(n + m)$. Also, there exist $O(n + m)$ time algorithms based on BFS for the following problems:
  - Testing whether G is connected.
  - Computing a spanning tree of G
  - Computing the connected components of G
  - Computing, for every vertex v of G, the minimum number of edges of any path between s and v.