AK/COSC 3432.03

Final Examination (Take Home)

Date: March 29, 1999, 7:00pm—in class

Due date/time: April 5, 1999, by 4:00pm—in Rm522, Atkinson

10 MARKS **(A)** For the (non-optimal) solution of the max/min problem we had arrived in class at a timing function (which counts the number of comparisons) $f(n)$ such that

$$f(1) = 0$$
$$f(2) = 1$$

and for $n > 2$

$$f(n) = f(\left\lceil \frac{n}{2} \right\rceil) + f(\left\lfloor \frac{n}{2} \right\rfloor) + 2$$

(i) Show that it leads to the recurrence

$$g(n) = \begin{cases} 1 & n = 2 \\ 2 & n = 3 \\ g(\lceil \frac{n}{2} \rceil) & n > 3 \end{cases} \tag{1}$$

where $g(n) = f(n) - f(n-1)$.

(ii) **Solve (1) in closed form *exactly*, and through it find $f(n)$, also *exactly*.**

Hints. Solving (1) you find that the recursion sometimes breaks off at $n = 2$ sometimes at $n = 3$. Note that it is *not* allowed to do $g(3) = g(\lceil 3/2 \rceil) = g(2) = 1$ because $g(3)$ is a *basis* case and its computation must not use the recursive formula which is valid *only* when $n > 3$.

You are looking for a final answer for $f(n)$ as follows.

$$f(n) = 3 \times 2^{\lfloor \log n \rfloor - 1} - 2 + \begin{cases} 2(n - 2^{\lfloor \log n \rfloor}) & n \leq 3 \times 2^{\lfloor \log n \rfloor - 1} \\ n - 2^{\lfloor \log n \rfloor - 1} & \textbf{otherwise} \end{cases}$$

where all log's are base-2. To do this, observe that (in binary) numbers n have the possible forms

$$10\overbrace{0\cdots0}^{\text{all 0s}}$$

$$10\overbrace{\cdots}^{\text{some 1s}}$$

$$11\overbrace{0\cdots0}^{\text{all 0s}}$$

$$11\overbrace{\cdots}^{\text{some 1s}}$$

and determine which such n break off the recursion (1) at 2 and which at 3.

A proof by induction of the above answer is unacceptable. The answer must be calculated.

5 MARKS **(B)** The *Euclidean Algorithm* for finding the $\gcd(a, b)$ ($a \geq b$ natural numbers, not both 0) is captured by

$$\gcd(a, 0) = a$$
$$\gcd(a, b) = \gcd(b, a \mod b)$$

where "$a \mod b$" is the remainder of the division of a by b.

Prove that this algorithm terminates in less than $2 \log_2 b$ divisions.

5 MARKS **(C)** Using generating functions, **but no other way**, show that $\sum_{i=0}^{n} i = n(n+1)/2$.

5 MARKS **(D)** Prove the *lower bound of* $n - 1$ *comparisons* to find the maximum of n distinct elements using a "states and wrestlers" argument.

Specifically, you are required to use the state (u, w, l), where *at any moment in time*, u, the "uninitiated", is the number of wrestlers that never fought, w is the number of wrestlers that never lost, and l is the number of wrestlers that lost *at least once*.

5 MARKS **(E)** Recall the recursive algorithm that exponentiates, i.e., finds x^n for a natural number n, in $O(\log n)$ multiplications.

Its timing $T(n)$, i.e., number of multiplications as a function of the value n of the exponent, is given by

$$T(n) = \begin{cases} 0 & \textbf{if } n = 1 \\ T(\lfloor n/2 \rfloor) + 1 + (n \mod 2) & \textbf{if } n > 1 \end{cases} \tag{1}$$

Solve (1) *exactly*, i.e., *not* in O-notation.

5 MARKS **(F)**

(a) Compute 2^{-1} in $\mathbb{I}_{15}$.

(b) Does 3^{-1} exist in $\mathbb{I}_{15}$? If "yes", which integer is it, if "no", why?