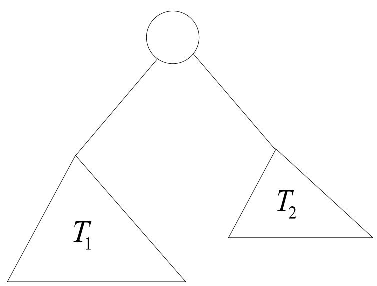
An application of the convolution to counting binary trees

1. The problem

We want to find the number of all extended trees of n internal nodes.

Let the sought quantity be called x_n.

Refer to the following figure, where the tree has n internal nodes, while subtree T_1 has m internal nodes and subtree T_2 has r internal nodes.



Thus, $n = m + r + 1$. We can choose T_1 in x_m different ways, and for each way, we can have x_r different versions of T_2. And that is true for each size of T_1. Thus, the recurrence equations for x_n are

$$x_0 = 1 \text{ (there is only one empty tree)} \tag{1}$$

$$x_n = \sum_{n-1=m+r} x_m x_r, \text{ for } n > 0 \tag{2}$$

2. The solution of (1) and (2)

We recognize in (2) the convolution resulting from $G(z)^2$, where

$$G(z) = x_0 + x_1 z + x_2 z^2 + \cdots + x_n z^n + \cdots \tag{3}$$

Indeed,

$$\begin{aligned} G(z)^2 &= x_0^2 + (x_0 x_1 + x_1 x_0)z + \cdots + (\sum_{n-1=m+r} x_m x_r) z^{n-1} + \cdots \\ &= x_1 + x_2 z + \cdots + x_n z^{n-1} + \cdots \end{aligned} \tag{4}$$

Thus, $zG(z)^2 + x_0 = G(z)$, or

$$zG(z)^2 - G(z) + 1 = 0 \tag{5}$$

We solve (5) for $G(z)$ to find

$$G(z) = \begin{cases} \dfrac{1+\sqrt{1-4z}}{2z} & \text{, or} \\ \dfrac{1-\sqrt{1-4z}}{2z} \end{cases} \tag{6}$$

equivalently,

$$zG(z) = \begin{cases} \dfrac{1+\sqrt{1-4z}}{2} & \text{, or} \\ \dfrac{1-\sqrt{1-4z}}{2} \end{cases} \tag{6$'$}$$

The first of (6′) is false for $z = 0$, so we keep and develop the second solution. To this end we expand $\sqrt{1-4z} = (1-4z)^{1/2}$ by the binomial expansion.

$$(1-4z)^{1/2} = 1 + \cdots + \binom{1/2}{n}(-4z)^n + \cdots$$

Let us work with the coefficient $\binom{1/2}{n}(-4)^n$.

$$\begin{aligned} \binom{1/2}{n}(-4)^n &= \frac{1/2(1/2-1)(1/2-2)\ldots(1/2-[n-1])}{n!}(-4)^n \\ &= (-1)^n 2^{2n} \frac{1(1-2\cdot 1)(1-2\cdot 2)\ldots(1-2\cdot[n-1])}{2^n n!} \\ &= (-1)^n(-1)^{n-1}2^n \frac{(2\cdot 1-1)(2\cdot 2-1)\ldots(2\cdot[n-1]-1)}{n!} && (7) \\ &= -2\frac{(2\cdot 1-1)[2\cdot 1](2\cdot 2-1)[2\cdot 2]\ldots(2\cdot[n-1]-1)[2\cdot[n-1]]}{n!(n-1)!} && (8) \\ &= -2\frac{(2n-2)!}{n!(n-1)!} \\ &= -\frac{2}{n}\binom{2n-2}{n-1} \end{aligned}$$

Going from (7) to (8) above we introduced factors $[2 \cdot 1], [2 \cdot 2], \ldots, [2 \cdot [n-1]]$ in order to "close the gap" and make the numerator be a factorial. This has spent $n-1$ of the n 2-factors in 2^n, but introduced $(n-1)!$ on the numerator, hence we balanced it out in the denominator.

It follows that, according to the second case of (6),

$$x_n = G(z)[z^n] = \frac{1 - \left(1 - 2z - \cdots - \dfrac{2}{n}\dbinom{2n-2}{n-1} z^n - \cdots\right)}{2z}[z^n] \tag{9}$$

where for any generating function $G(z)$, $G(z)[z^n]$ denotes the coefficient of z^n.

In short,

$$x_n = \frac{1}{n+1}\binom{2n}{n}$$

Don't forget that we want $G(z)[z^n]$; we have adjusted for the division by z.