CS3432.03 (W99)

Problem Set # 2

1. Themes: Divide and Conquer; Recurrence Relations; Generating Functions

1. Find the *average* number of comparisons taken to build a *sort-tree* of n nodes under the "randomness assumption" that it is equally likely that the root is the k-th largest, $k = 1, 2, \ldots, n$.

Hint. The notion of *sort-tree* is known from your Data Structures course—note that we are talking about *any* sort-tree, *not* just AVL types.

Prove that if $T(n)$ denotes the average number of comparisons, then we have a recurrence

$$T(1) = 0$$
$$T(n) = n - 1 + \frac{2}{n}\sum_{k=1}^{n-1} T(k)$$

Solve the recurrence in closed form.

2. Do problem 6, p.226.

3. Prove that finding "max" and "2nd max" of n distinct elements can be done in $\le n + \lceil \log n \rceil - 2$ *comparisons* even in the case that n is not a power of 2.

Hint. Prove that Algorithm 3.3 (p.176 of the text) works with the simple modification that we use *explicit linking* and no swapping for any n, but, of course, the trees that it builds are not binomial anymore.

Find the recurrence (it involves both $\lceil \cdots \rceil$ and $\lfloor \cdots \rfloor$) that governs the number of comparisons and solve it in closed form to prove that the tree built for any n is built in exactly $n - 1$ comparisons.

Finally prove that any such tree has $\le \lceil \log n \rceil$ nodes at level 1 (the root is at level 0).

4. Pretend that two-way (recursive) merge-sort splits an array of size n into two parts of size $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$, sorts them, and then merges the resulting "halves" in linear time.

Set up the recurrence relation for this problem, and solve it *exactly*.

5. One can compute $\sum_{i=0}^{n-1} \binom{i}{k-1}$ in closed form, for $1 \le k \le n$, using the identity $\binom{i}{k-1} = \binom{i+1}{k} - \binom{i}{k}$.

In the present problem ignore all properties of the symbol $\binom{n}{i}$ except that it is the coefficient of $a^i b^{n-i}$ in $(a + b)^n$, and *using generating functions* find $\sum_{i=0}^{n-1} \binom{i}{k-1}$.