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EECS 1028 Z. Problem Set No2 —SOLUTIONS

Posted: Feb. 19, 2024

The concept of "late assignments" does not exist in this course, as you recall.

1. (3 MARKS) Give an example of two equivalence relations R and S on the set $A = \{1, 2, 3\}$ such that $R \cup S$ is *not* an equivalence relation.

Proof. It is obvious that $R \cup S$ will inherit trivially reflexivity and symmetry. Let us find specific R and S whose union will fail transitivity.

Take $R = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$ and $S = \{(1,1),(2,2),(3,3),(2,3),(3,2)\}$. Clearly, each is an equivalence relation on A.

But $\{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2)\}$ is not. At least $(1,3)$ is needed for transitivity. □

2. (3 MARKS) Let P be a reflexive relation on A that satisfies $aPb \wedge aPc \to bPc$. Prove that P is an equivalence relation on A.

Caution. This $aPb \wedge aPc \to bPc$ is <u>not</u> exactly transitivity!

Proof.

(a) *Symmetry*: Let aPb. By reflexivity I have aPa. By the rule given, $aPb \wedge aPa \to bPa$.

(b) *Transitivity*: Let aPb and bPc, hence (by symm.), $bP\underline{a} \wedge bP\underline{c}$. By the rule given we get $\underline{a}P\underline{c}$. □

3. (3 MARKS) Show for a relation $\mathbb{S}$ that if both the range and the domain are sets, then $\mathbb{S}$ is a set.

Proof. We know that $\mathbb{S} \subseteq \overbrace{\underbrace{\mathrm{dom}(\mathbb{S})}_{set} \times \underbrace{\mathrm{ran}(\mathbb{S})}_{set}}^{set}$. From the diagram to the left, and the subclass theorem, we are done. □

4. (3 MARKS) Let $A \neq \emptyset$ be a set. Prove that A^2 is an equivalence relation on A.

Proof. Since we did not say that A is a relation, the exponent ^{2} refers to $A \times A$.

Given that

$$A^2 = \{(x, y) : x \in A \wedge y \in A\} \tag{1}$$

we have

(a) *Reflexive*: Indeed, if $a \in A$ then also $a \in A$, hence $(x, x) \in A^2$.

(b) *Symmetric*: Say $(x, y) \in A^2$. Then $x \in A$ and $y \in A$. Swapping the roles of x and y and looking at (1) we have $(y, x) \in A^2$ as well.

(c) *Transitive*: Let

$$(x, y) \in A^2 \tag{2}$$

and

$$(y, z) \in A^2 \tag{3}$$

This entails —*IN PARTICULAR!*— that $x \in A$ (by (2)) and $z \in A$ (by (3)). Thus $(x, z) \in A^2$ (by (1)). □

5. (4 MARKS) Let R be symmetric. Show that so is R^n for the arbitrary $n > 0$.

Hint. No need for induction. Show this by noting (from class that) $R^n = \overbrace{R \circ \cdots \circ R}^{n\ R}$.

Proof. Let R be symmetric, and also let xR^ny. The latter means that we have

$$xRa_1Ra_2 \cdots Ra_{n-2}Ra_{n-1}Ry, \text{ for some } a_i \tag{1}$$

By the red assumption, (1) can be written backwards as

$$yRa_{n-1}Ra_{n-2} \cdots Ra_2Ra_1Rx, \text{ for the same } a_i \text{ as in (1)} \tag{2}$$

(2) says yR^nx. Done. □

6. (3 MARKS) Show that a relation $\mathbb{R}$ is symmetric iff $\mathbb{R} = \mathbb{R}^{-1}$.

Caution. There are two directions here.

Proof.

(a) Let $\mathbb{R}$ be symmetric. Prove that $\mathbb{R} = \mathbb{R}^{-1}$.

$$x\mathbb{R}^{-1}y \overset{def\ of\ inverse}{\iff} y\mathbb{R}x \overset{red\ hyp}{\iff}{}^{\dagger} x\mathbb{R}y$$

So $\mathbb{R}^{-1} = \mathbb{R}$.

(b) Let $\mathbb{R}^{-1} = \mathbb{R}$. I will prove that $\mathbb{R}$ is symmetric, that is, $\boxed{x\mathbb{R}y \equiv y\mathbb{R}x}$:

$$x\mathbb{R}y \overset{red\ Let}{\iff} x\mathbb{R}^{-1}y \overset{def\ inver.}{\iff} y\mathbb{R}x$$

□

7. (3 MARKS) Show that if a relation $\mathbb{S}$ is transitive, then so is $\mathbb{S}^{-1}$.

Proof. Assume $a\mathbb{S}^{-1}b \wedge b\mathbb{S}^{-1}c$.

The above says (I swapped order for visual ease)

$$c\mathbb{S}b \wedge b\mathbb{S}a \tag{1}$$

By transitivity of $\mathbb{S}$ I get

$$c\mathbb{S}a$$

from (1). But that says $a\mathbb{S}^{-1}c$. By the red assumption we just proved transitivity of $\mathbb{S}^{-1}$. □

[†] I remarked in class more than once that the unidirectional definition of symmetry, "$x\mathbb{R}y \to y\mathbb{R}x$" is really an equivalence "$x\mathbb{R}y \equiv y\mathbb{R}x$" as we get the $\leftarrow$ direction by repeating the definition right-to-left.

8. (5 MARKS) Let R on A be reflexive and symmetric. Prove that R^+ is an equivalence relation.

Proof. Since R^+ is transitive we will focus on proving reflexivity and symmetry.

(a) *Reflexivity*. Since

$$R^+ = \bigcup_{n=1}^{\infty} R^n \tag{†}$$

and as we observe —by (†) or straight from the definition of transitive closure— that we have

$$R \subseteq R^+ \tag{1}$$

we are done by reflexivity of R: **If $a \in A$, then aRa hence aR^+a by (1)**.

(b) *Symmetry*. Let aR^+b. By (†), we have,

$$\text{for some } n > 0,\ aR^nb \tag{2}$$

By Problem 5, we have bR^na, hence (def. of $\bigcup$)

$$b \bigcup_{n=1}^{\infty} R^n a$$

By the above and (†), we have

$$bR^+a$$

Given the red Let, what we concluded immediately above is what we needed to establish symmetry. □