

York University

Faculty of Science and Engineering

MATH 1090: Facts-List for the December 2009 Examination (to be held Dec 19, 2009)

The following are the axioms of Propositional Calculus: In what follows, A, B, C stand for arbitrary formulae.

Properties of \equiv

$$\text{Associativity of } \equiv \quad ((A \equiv B) \equiv C) \equiv (A \equiv (B \equiv C)) \quad (1)$$

$$\text{Symmetry of } \equiv \quad (A \equiv B) \equiv (B \equiv A) \quad (2)$$

Properties of \perp, \top

$$\top \text{ vs. } \perp \quad \top \equiv \perp \equiv \perp \quad (3)$$

Properties of \neg

$$\text{Introduction of } \neg \quad \neg A \equiv A \equiv \perp \quad (4)$$

Properties of \vee

$$\text{Associativity of } \vee \quad (A \vee B) \vee C \equiv A \vee (B \vee C) \quad (5)$$

$$\text{Symmetry of } \vee \quad A \vee B \equiv B \vee A \quad (6)$$

$$\text{Idempotency of } \vee \quad A \vee A \equiv A \quad (7)$$

$$\text{Distributivity of } \vee \text{ over } \equiv \quad A \vee (B \equiv C) \equiv A \vee B \equiv A \vee C \quad (8)$$

$$\text{Excluded Middle} \quad A \vee \neg A \quad (9)$$

Properties of \wedge

$$\text{Golden Rule} \quad A \wedge B \equiv A \equiv B \equiv A \vee B \quad (10)$$

Properties of \rightarrow

$$\text{Implication} \quad A \rightarrow B \equiv A \vee B \equiv B \quad (11)$$

The Primary Boolean rules are:

$$\frac{A, A \equiv B}{B} \quad (Eqn)$$

and

$$\frac{A \equiv B}{C[\mathbf{p} := A] \equiv C[\mathbf{p} := B]} \quad (Leib)$$

The following are the Predicate Calculus Axioms:

Any partial generalisation of any formula in groups Ax1–Ax6 is an axiom for Predicate Calculus.

Groups **Ax1–Ax6** contain the following schemata:

Ax1. Every tautology.

Ax2. $(\forall \mathbf{x})A \rightarrow A[\mathbf{x} := t]$, for any term t .

Ax3. $A \rightarrow (\forall \mathbf{x})A$, provided \mathbf{x} is not free in A .

Ax4. $(\forall \mathbf{x})(A \rightarrow B) \rightarrow (\forall \mathbf{x})A \rightarrow (\forall \mathbf{x})B$.

Ax5. For *each* object variable \mathbf{x} , the formula $\mathbf{x} = \mathbf{x}$.

Ax6. For any terms t, s , the schema $t = s \rightarrow (A[\mathbf{x} := t] \equiv A[\mathbf{x} := s])$.

The following metatheorems are good for **both** Propositional and Predicate Calculus:

1. *Redundant True.* $\Gamma \vdash A$ iff $\Gamma \vdash A \equiv \top$

2. *Modus Ponens (MP).* $A, A \rightarrow B \vdash B$

3. *Cut Rule.* $A \vee B, \neg A \vee C \vdash B \vee C$

4. *Deduction Theorem.* If $\Gamma, A \vdash B$, then $\Gamma \vdash A \rightarrow B$

5. *Proof by contradiction.* $\Gamma, \neg A \vdash \perp$ iff $\Gamma \vdash A$

6. *Post's Theorem.* (Also called “tautology theorem”, or even “completeness of Propositional Calculus theorem”)

If $\models_{\text{taut}} A$, then $\vdash A$.

Also: If $\Gamma \models_{\text{taut}} A$, then $\Gamma \vdash A$.

7. *Proof by cases.* $A \rightarrow B, C \rightarrow D \vdash A \vee C \rightarrow B \vee D$

Also the special case: $A \rightarrow B, C \rightarrow B \vdash A \vee C \rightarrow B$

Translations

$(\exists \mathbf{x})A$ translates to $\neg(\forall \mathbf{x})\neg A$

$(\forall \mathbf{x})_B A$ translates to $(\forall \mathbf{x})(B \rightarrow A)$

$(\exists \mathbf{x})_B A$ translates to $(\exists \mathbf{x})(B \wedge A)$

Useful facts from Predicate Calculus (proved in class—you may use them without proof):

We **know** that SL and WL are **derived rules** useful in equational proofs within predicate calculus.

► More “rules” and (meta)theorems.

(i) *Dummy renaming.*

If \mathbf{z} does not occur in $(\forall \mathbf{x})A$ as either free or bound, then $\vdash (\forall \mathbf{x})A \equiv (\forall \mathbf{z})(A[\mathbf{x} := \mathbf{z}])$

If \mathbf{z} does not occur in $(\exists \mathbf{x})A$ as either free or bound, then $\vdash (\exists \mathbf{x})A \equiv (\exists \mathbf{z})(A[\mathbf{x} := \mathbf{z}])$

(ii) \forall over \circ distribution, where “ \circ ” is “ \vee ” or “ \rightarrow ”.

$\vdash A \circ (\forall \mathbf{x})B \equiv (\forall \mathbf{x})(A \circ B)$, **provided** \mathbf{x} is not free in A

\exists over \wedge distribution

$\vdash A \wedge (\exists \mathbf{x})B \equiv (\exists \mathbf{x})(A \wedge B)$, **provided** \mathbf{x} is not free in A

(iii) \forall over \wedge distribution.

$\vdash (\forall \mathbf{x})A \wedge (\forall \mathbf{x})B \equiv (\forall \mathbf{x})(A \wedge B)$

\exists over \vee distribution.

$\vdash (\exists \mathbf{x})A \vee (\exists \mathbf{x})B \equiv (\exists \mathbf{x})(A \vee B)$

(iv) \forall commutativity (symmetry).

$\vdash (\forall \mathbf{x})(\forall \mathbf{y})A \equiv (\forall \mathbf{y})(\forall \mathbf{x})A$

\exists commutativity (symmetry).

$\vdash (\exists \mathbf{x})(\exists \mathbf{y})A \equiv (\exists \mathbf{y})(\exists \mathbf{x})A$

(v) *Specialisation. “Spec”* $(\forall \mathbf{x})A \vdash A[\mathbf{x} := t]$, for any term t .

Dual of Specialisation. $A[\mathbf{x} := t] \vdash (\exists \mathbf{x})A$, for any term t .

(vi) *Generalisation. “Gen”* If $\Gamma \vdash A$ and if, moreover, the formulae in Γ have **no free \mathbf{x} occurrences**, then also $\Gamma \vdash (\forall \mathbf{x})A$.

(vii) \forall *Monotonicity.* If $\Gamma \vdash A \rightarrow B$ so that the formulae in Γ have **no free \mathbf{x} occurrences**, then we can infer

$\Gamma \vdash (\forall \mathbf{x})A \rightarrow (\forall \mathbf{x})B$

- (viii) \forall Introduction; a special case of \forall Monotonicity. If $\Gamma \vdash A \rightarrow B$ so that neither the formulae in Γ nor A have **any free x occurrences**, then we can infer

$$\Gamma \vdash A \rightarrow (\forall x)B$$

- (ix) Finally, the *Auxiliary Variable (“witness”) Metatheorem*. If $\Gamma \vdash (\exists x)A$, and if y is a variable that **does not** occur as either free or bound variable in any of $(\exists x)A$ or B or the formulae of Γ , then

$$\Gamma, A[x := y] \vdash B \text{ implies } \Gamma \vdash B$$

Semantics facts

Propositional Calculus	Predicate Calculus
(Boolean Soundness) $\vdash A$ implies $\models_{\text{taut}} A$ (Post) $\models_{\text{taut}} A$ implies $\vdash A$	$\vdash A$ does NOT imply $\models_{\text{taut}} A$ However, (Post) $\models_{\text{taut}} A$ implies $\vdash A$ (Pred. Calc. Soundness) $\vdash A$ implies $\models A$ (Gödel Completeness) $\models A$ implies $\vdash A$



CAUTION! The above facts/tools are only a fraction of what we have covered in class. They are *very important and very useful*, and that is why they are listed for your reference here.

You can also use *without proof* **ALL** the things we have covered (such as the absolute theorems known as “ \exists -definition”, “de Morgan’s laws”, etc.).

But these —the unlisted ones— are up to you to remember and to correctly state!

*Whenever in doubt of whether or not a “tool” you are about to use was indeed covered in class, **prove** the validity/fitness of the tool before using it!*

