Department of EECS

MATH1090 A. Problem Set No1 —SOLUTIONS
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1. (3 MARKS) Prove that $)p($ is *NOT* a wff.

Hint. One way to prove this is to analyse formula-constructions/calculations. The other is to look at the inductive definition of formulas: can it be applied to define "$)p($" as a wff? Why?

Proof. I give two solutions, one for each Hint above.

(a) **Answer 1**. Look at formula calculations.

Well, IF "$)p($" is a wff, then it MUST appear in some formula calculation.

But NO string we write in a formula calculation can start with "$)$". They ALL start with one of $\mathbf{p}, \bot, \top$ or "$($".

So "$)p($" being unable to appear in a formula calculation is NOT a wff.

OR

(b) **Answer 2**. Look at the inductive definition:

A wff A is one of:

i. Atomic: $\mathbf{p}, \bot, \top$
ii. $(\neg B)$, where B is a wff
iii. $(B \circ C)$, where B, C are wff

None of the cases above starts with a "$)$" so "$)p($" is NOT a wff.

This second proof essentially re-proves and uses that "a wff begins with an atomic wff or with a $($". It is OK to use this quoted reason without proof.

□

2. (1 MARK) Let Q, P be wff's. Prove that so is

$$(((P \to Q) \to P) \to P) \tag{1}$$

Proof. This is not about truth or falsehood of the above. It is just to verify that *its syntax is right*!

OK, <u>ONE way is to so prove by a formula calculation</u>. It is given that P and Q are wff, so each appears in some formula calculation. By concatenating the two constructions we obtain a (syntactically) valid formula construction since checking the validity of such a <u>constructions</u> for each of the P and Q <u>separately</u> checks OK; so if I recheck after I write them consecutively <u>nothing in the checking will change</u>!

$$\boxed{\ldots P \ldots}\boxed{\ldots Q \ldots} \tag{2}$$

We continue the formula calculation (2) below until we can include (1)!

$$\boxed{\ldots P \ldots}\boxed{\ldots Q \ldots}\overbrace{(P \to Q)}^{glue\ P\ \&\ Q}\overbrace{((P \to Q) \to P}^{glue\ (P \to Q)\ \&\ P}\overbrace{(((P \to Q) \to P) \to P)}^{glue\ ((P \to Q) \to P)\ \&\ P}$$

The above is a *valid formula calculation* since in the *three added steps* I glued strings I already had to my *left*.

Since $(((P \to Q) \to P) \to P)$ appears in the above formula calculation it is a wff.

The OTHER way to do this exercise is to use the *recursive wff definition* (see 1b above). By iii. in said definition,

- since P and Q are wff, then so is $(P \to Q)$.
- since $(P \to Q)$ and P are wff, then so is $((P \to Q) \to P)$.
- since $((P \to Q) \to P)$ and P are wff, so is $(((P \to Q) \to P) \to P)$.

$\square$

3. (6 MARKS) Recall that a **schema** is a tautology iff *all* its *instances* are tautologies.

Which of the following six schemata are tautologies? Show the whole process that led to your answers, *including truth tables or equivalent short cuts, if you used one or the other,* <u>*and words of explanation if needed*</u>.

I note that in the six sub-questions below I am <u>NOT</u> using <u>all</u> the <u>formally necessary</u> brackets. You need to reinsert missing brackets to answer correctly.

- $A \to B \equiv \neg A \vee B$

  Answer. By the priorities of glue, the above says

$$(A \to B) \equiv (\neg A \vee B)$$

  Let me use a truth table.

A	B	A	$\to$	B	$\equiv$	$\neg$	A	$\vee$	B
f	**f**		**t**		**t**	**t**		**t**	
f	**t**		**t**		**t**	**t**		**t**	
t	**f**		**f**		**t**	**f**		**f**	
t	**t**		**t**		**t**	**f**		**t**	

□

- $p \to q \to (p \to \bot) \vee q$

  Answer. This says (missing essential brackets inserted)

$$p \to \Big(q \to (p \to \bot) \vee q\Big) \quad (1)$$

  I use shortcuts according to the truth value of p. So pick a state s.

  (a) Case where p is **f** in s (that is, $s(p) = \mathbf{f}$). Thus (1) is true no matter what is the value of $\Big(q \to (p \to \bot) \vee q\Big)$.

  (b) Case where p is **t** in s.
  Subcases according to the $s(q)$-value.

  i. $s(q) = \mathbf{f}$. Then —no matter what $\overline{s}\Big((p \to \bot) \vee q\Big)$ is— (1) evaluates **t** by the truth table for "$\to$" (case "**t**$\to$**t**").

  ii. $s(q) = \mathbf{t}$. Then $\overline{s}\Big((p \to \bot) \vee q\Big) = \mathbf{t}$ by the "$\vee q$" part. Thus the big bracket in (a) is **t** ($\mathbf{t} \to \mathbf{t}$ case of truth table) and thus (1) evaluate as **t**, again due to the truth table due to the "**t**$\to$**t**" case of truth table.

□

- $A \wedge B \equiv A \equiv B$

  Answer. By the priorities of glue, the absolutely minimum number of necessary brackets make the formula into

$$A \wedge B \equiv (A \equiv B)$$

Not a tautology schema. If this were a tautology for any wff A and B then would be so for A being just p and B being just q.
But

$$p \wedge q \equiv (p \equiv q)$$

is not a tautology. In the state s where $s(p) = s(q) = \mathbf{f}$ we have the lhs of $\equiv$ false but the rhs is true.

DO NOT SAY "if both A and B are $\mathbf{f}$, then the formula is false." Bad! What if A and B are NEVER BOTH FALSE? We Don't KNOW what A and B might be!!

That is why you MUST look at a special case where you have control: VARIABLES!

□

- $A \wedge B \to C \equiv B \to A \to C$

  Answer. The formula with some missing brackets restored is

$$A \wedge B \to C \equiv B \to (A \to C) \tag{1}$$

Let us answer by truth table shortcuts.

Cases according to truth value of B

(a) B evaluates as $\mathbf{f}$. Then by the truth table for $\to$ (and $\wedge$), both sides of "$\equiv$" evaluate to $\mathbf{t}$. Good!

(b) B evaluates as $\mathbf{t}$. Then we have subcases by truth values of A:

   i. A evaluates as $\mathbf{f}$. Then seeing that $A \to C$ evaluates as true, we have at once that both sides of "$\equiv$" evaluate to $\mathbf{t}$. Good!

   ii. A evaluates as $\mathbf{t}$. We will need subcases according to the value of C:

      1. C evaluates $\mathbf{t}$. Then seeing that $A \to C$ evaluates as true, we have at once —tables for $\to$— that both sides of "$\equiv$" evaluate to $\mathbf{t}$. Good!

2. <u>C evaluates $\mathbf{f}$</u>. Then seeing that $A \to C$ evaluates as false, we have at once —tables for $\to$— that both sides of "$\equiv$" evaluate to $\mathbf{f}$. Good!

$\square$

- $(A \to B) \to \neg B \to \neg A$

  This means

$$(A \to B) \to (\neg B \to \neg A)$$

Answer. Let us use truth tables to show it is a tautology (schema).

A	B	A	$\to$	B	$\to$	$\neg$	B	$\to$	$\neg$	A
f	**f**		**t**		t	**t**		**t**	**t**	
f	**t**		**t**		t	**f**		**t**	**t**	
t	**f**		**f**		t	**t**		**f**	**f**	
t	**t**		**t**		t	**f**		**t**	**f**	

$\square$

- $A \wedge B \to C \equiv (B \to A) \to C$

  Anwser. NOT a tautology schema because the instant

$$p \wedge q \to r \equiv (p \to q) \to r \tag{1}$$

is not a tautology:
Indeed pick a state v such that

$$v(p) = v(q) = v(r) = \mathbf{f}$$

Then by the truth tables for $\to$ the lhs of $\equiv$ evaluates $\mathbf{t}$ but the rhs evaluates $\mathbf{f}$. $\square$

4. (2 MARKS) Prove for all wff A, B that we have $A \wedge \neg A \models_{\text{taut}} B$.

Proof. There is no state that makes the lhs of $\models_{true}$ <u>all true</u>. Thus the statement is correct *without us having to do any work*. *Check definition!* $\square$

5. (6 MARKS) By using truth tables, or using related shortcuts, examine whether or not the following tautological implications are correct.

In order to show that a tautological implication that involves *meta*-variables for formulas —i.e., it is a schema— is *incorrect* you *must* consider a special case that *is* incorrect (since some other special cases might *work*).

Show the whole process that led to each of your answers.

- $A \vee \neg A \models_{\text{taut}} \top$
  Correct. Both sides are always **t**. □
- $\top \models_{\text{taut}} A \wedge B$
  INcorrect. Here is an instant that does not go through:
  $$\top \models_{\text{taut}} p \wedge q$$
  Take s such that $s(p) = \mathbf{f}$. YET, in this state lhs is true (but rhs is **f**)! □
- $\bot \models_{\text{taut}} A \wedge B$
  Correct. No state makes the lhs **t** so we have OK without lifting a finger. □
- $A, A \rightarrow B \models_{\text{taut}} B$
  Correct. Imagine the lhs all true. So A is and so is $A \rightarrow B$. By truth table of $\rightarrow$, the rhs is **t**. □
- $A \equiv B \models_{\text{taut}} B \rightarrow A$
  Correct. Imagine any state where lhs evaluates true.
  The only cases of such outcomes must be **f**, **f** or **t**, **t**.
  Under both cases the rhs is **t** by the tables for $\rightarrow$.
- $A \vee B \models_{\text{taut}} B \wedge A \equiv A \equiv B$
  Correct. Think of rhs as $B \wedge A \equiv (A \equiv B)$.
  Let now lhs evaluate **t**. Then *at least one* of A or B is **t**
  Cases:
  (a) Actually both are true. Then $A \wedge B$ is **t** and so is $A \equiv B$. Thus rhs is **t**.

(b) Only one of A and B is **t**. Then $A \wedge B$ is **f** and so is $A \equiv B$. Thus rhs is **t** again! $\square$

6. (6 MARKS) Write down the most simplified result of the following substitutions, *whenever the requested substitution makes sense*. Whenever a requested substitution does not make sense, explain exactly why it does not.

Show the whole process that led to each of your answers in each case.

Remember the priorities of the various connectives as well as that of the meta-expression "$[\mathbf{p} := \ldots]$"! The following formulas have not been written with all the formally required brackets.

- $(q \to p)[p := r]$
  Answer. $(q \to r)$ $\square$
- $(q \to p)[p' := r]$
  Answer. $(q \to p)$ —no change! $\square$
- $p \to \top[\top := \bot]$
  Answer. ILLEGAL! $\top$ is *NOT* a variable! $\square$
- $p \to \top[p := \mathbf{f}]$
  Answer. ILLEGAL! **f** is *NOT* a *wff*! $\square$
- $(\bot \to r \to q)[q \wedge r := p]$
  Answer. ILLEGAL! $q \wedge r$ is *NOT* a variable! $\square$
- $(\bot \to r \to q)[q := r][r := p]$
  Answer. The above means

$$\Big((\bot \to r \to q)[q := r]\Big)[r := p]$$

  So we get $(\bot \to p \to p)$

$\square$