Lassonde School of Engineering

Dept. of EECS
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MATH1090 A. Problem Set No2 —SOLUTIONS

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It is not allowed to use truth tables (or any of their shortcuts) in ANY of the problems below. Such methods get zero marks.

1. By definition, in a Σ-proof we are free to write an axiom A ($A \in \Lambda$) or a "hyp" A from Σ ($A \in \Sigma$) as many times as we like. Each time the justification is "axiom" or "wff from Σ" according to the case.

(a) (2 MARKS) Can we also write, say, *consecutively 10 times* in a row the *result* B of Eqn applied on *previous* wff X and Y in the proof? What reason will we give each of the 10 times?

Answer. Yes. Say the X and Y are on lines (i) and (j).

The same justification/annotation will be given in *each* of the 10 cases: $\langle (i) + (j) + \text{Eqn}\rangle$. $\square$

(b) (1 MARKS) What if the 10 times are not consecutive? Can we do it? What reason will we give?

Answer. Yes, we can do it exactly as above. Say the X and Y are on lines (i) and (j).

The same justification/annotation will be given in *each* of the 10 not consecutive cases: $\langle (i) + (j) + \text{Eqn}\rangle$. $\square$

2. (4 MARKS) Prove **Equationally** that $A, B \vdash A \equiv B$.

Proof.

$A \equiv B$
$\Leftrightarrow \langle$Leib + Red. $\top$ META: $A, B \vdash A \equiv \top$; Denom $\mathbf{p} \equiv B\rangle$
$\top \equiv B$
$\Leftrightarrow \langle$Red. $\top$ THM$\rangle$
B bingo!

□

3. (4 MARKS) Prove **Equationally** that for any A,

$$\bot \vdash A$$

Proof.

A
$\Leftrightarrow \langle$thm from class/Notes$\rangle$
$\bot \vee A$
$\Leftrightarrow \langle$Red. $\top$ META (i.e., $\bot \vdash \bot \equiv \top$) + Leib; Denom: $\mathbf{p} \vee A\rangle$
$\top \vee A$ bingo! thm from class/Notes

□

4. (4 MARKS) Prove **Equationally** that $\vdash A \wedge B \equiv B \wedge A$.

Hint. Insert the missing brackets first (but not the outermost).

Proof. This asks us to certify $\vdash (A \wedge B) \equiv (B \wedge A)$.

$A \wedge B$
$\Leftrightarrow \langle$GR$\rangle$
$A \vee B \equiv A \equiv B$
$\Leftrightarrow \langle$commute $\equiv$-chain$\rangle$
$A \vee B \equiv B \equiv A$
$\Leftrightarrow \langle$axiom + Leib; Denom: $\mathbf{p} \equiv B \equiv A\rangle$
$B \vee A \equiv B \equiv A$
$\Leftrightarrow \langle$GR$\rangle$
$B \wedge A$

□

5. (4 MARKS) Prove **Equationally** that $\vdash A \wedge (A \vee B) \equiv A$.

Proof.

$A \wedge (A \vee B)$
$\Leftrightarrow\langle$GR; I drop brackets because we MAY do so!$\rangle$
$A \vee A \vee B \equiv A \equiv A \vee B$
$\Leftrightarrow\langle$axiom + Leib; Denom: $\mathbf{p} \vee B \equiv A \equiv A \vee B\rangle$
$A \vee B \equiv A \equiv A \vee B$
$\Leftrightarrow\langle$commute an $\equiv$-chain$\rangle$
$A \vee B \equiv A \vee B \equiv A$
$\Leftrightarrow\langle\vdash X \equiv X$-thm + Red. $\top$ META + Leib; Denom: $\mathbf{p} \equiv A\rangle$
$\top \equiv A$
$\Leftrightarrow\langle$ Red. $\top$ THM$\rangle$
A

□

6. (3 MARKS) Prove **Equationally** that $\vdash B \wedge (A \vee \neg A) \equiv B$.

Proof.

$B \wedge (A \vee \neg A)$
$\Leftrightarrow\langle(A \vee \neg A)$-axiom + Red. $\top$ META + Leib; Denom: $B \wedge \mathbf{p}\rangle$
$B \wedge \top$
$\Leftrightarrow\langle$thm (class/Notes)$\rangle$
B

□

7. (3 MARKS) Prove **Equationally** that $\vdash A \lor B \lor \neg A$.

Proof.

$A \lor B \lor \neg A$
$\Leftrightarrow\langle$commute an $\lor$-chain$\rangle$
$A \lor \neg A \lor B$
$\Leftrightarrow\langle(A \lor \neg A)$-axiom + Red. $\top$ META + Leib; Denom: $\mathbf{p} \lor B\rangle$
$\top \lor B$ bingo!

□