York University
Department of Electrical Engineering and Computer Science
Lassonde School of Engineering

MATH 1090 A. FINAL EXAM —Solutions, December 19, 2023; 14:00-16:00

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Boolean Logic 1. (3 MARKS) Suppose $\Gamma \vdash A$ and $\Gamma \vdash B$. Does it follow that $\Gamma \vdash A \equiv B$?

If yes, give **an Equational proof**.

If not, use Boolean Soundness to justify your "NO".

Post's theorem is NOT allowed. Hilbert proof is NOT allowed.

Answer. YES.

Proof:

$$
\begin{aligned}
& A \equiv B \\
\Leftrightarrow & \langle \text{Leib} + \text{Red. } \top \text{ META } (\Gamma \vdash A \equiv \top); \text{ Denom: } \mathbf{p} \equiv B \rangle \\
& \top \equiv B \\
\Leftrightarrow & \langle \text{Red. } \top \text{ THM} \rangle \\
& B \qquad \text{Bingo!}
\end{aligned}
$$

□

Boolean Logic 2. (2 MARKS) If A is a wff, is (A) a wff too?

Prove the correctness of your answer using *formula calculations*.

Proof. Since A is a wff, there is a formula calculation

$\vdots$

A

for A.

Here is why this calculation CANNOT be extended to a formula calculation for (A):

There are TWO KINDS of steps possible where OVERALL BRACKETS ARE ADDED:

1. Using two previous STRINGS Q and R from the construction, OR
2. Using ONLY ONE previous STRING Q from the construction

Only case 2. above applies here (where "Q" is "A").

Case 2. with Q being A consists of the TWO sub-steps:

- must ADD the GLUE "$\neg$" in front of A

  and
- must ADD overall enclosing brackets around the result.

But "(A)" misses the "$\neg$".

So, I CANNOT continue the calculation of A to obtain (A), meaning I cannot obtain (A) by a formula calculation —the above sub-steps being the only ones AVAILABLE to do so.

In other words, (A) —being impossible to appear in a formula calculation— is NOT a wff. $\square$

Boolean Logic 3. (5 Marks) Prove **by Resolution**:

$$\vdash \Big((X \to Y) \to X\Big) \to X$$

Caution: **0 Marks** gained **if any other technique is used. In particular, Post's theorem is NOT allowed.**

A proof by resolution

1) MUST use a **graphical** proof by contradiction, and

2) *It cannot/must not* be "preloaded" with *a long Equational or Hilbert proof* only to conclude with *just ONE CUT*.

Such a proof, **IF correct**, loses half the points.

Proof. By DThm I prove instead

$$(X \to Y) \to X \vdash X$$

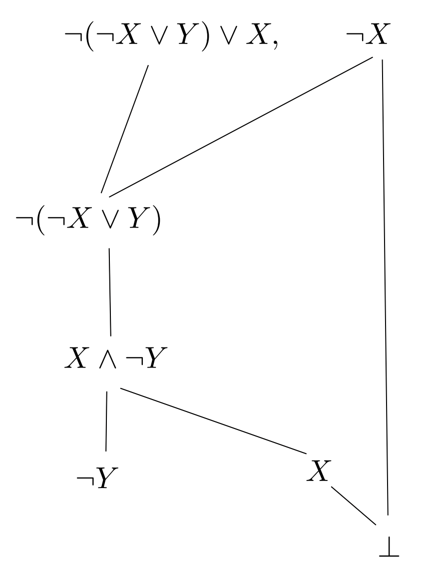
By **Proof By Contradiction** I will do instead

$$(X \to Y) \to X, \neg X \vdash \bot$$

or (via "$\neg\vee$" twice)

$$\neg(\neg X \vee Y) \vee X, \neg X \vdash \bot \tag{1}$$

I use resolution to prove (1):



□

Predicate Logic 1. (3 MARKS) **True** or **False** and **WHY —In the absence of a correct "WHY" the answer gets 0 MARKS**:

For any formula A, we have $\vdash (\forall \mathbf{z})(\forall \mathbf{x})(\forall \mathbf{z})(A \vee \neg A)$.

Answer. **True**: $A \vee \neg A$ is a schema from **Ax. 1** Group.

So EVERY partial Gen of it is an axiom, HENCE a theorem!

In particular, the partial Gen $(\forall \mathbf{z})(\forall \mathbf{x})(\forall \mathbf{z})(A \vee \neg A)$ is an axiom hence a theorem for all A!

ALTERNATIVELY, you may do this by a Hilbert proof, but the above is BETTER (indicates your deeper understanding of the Axioms).

1)	$A \vee \neg A$	$\langle$**Ax.** from Group 1$\rangle$
2)	$(\forall \mathbf{z})(A \vee \neg A)$	$\langle$1 + Gen; OK: NO hyp!$\rangle$
3)	$(\forall \mathbf{x})(\forall \mathbf{z})(A \vee \neg A)$	$\langle$2 + Gen; OK: NO hyp!$\rangle$
4)	$(\forall \mathbf{z})(\forall \mathbf{x})(\forall \mathbf{z})(A \vee \neg A)$	$\langle$3 + Gen; OK: NO hyp!$\rangle$

$\square$

Predicate Logic 2. (5 MARKS) Assume that $\mathbf{x}$ does not occur in t and that $A[\mathbf{x} := t]$ **is defined**.

Prove **Equationally** the ∃-version of the "**one-point rule**":

$$\vdash (\exists \mathbf{x})(\mathbf{x} = t \wedge A) \equiv A[\mathbf{x} := t]$$

Limitations:

- A **non-Equational** proof **will Max 0 points**.
- Equational proofs **will Max 3 points if the "⇔" symbol is omitted**.
- Properly annotate WL, if used: In particular, *you* **must** *check and* **acknowledge** that the hypothesis of the rule is an absolute theorem.

Proof. In the proof that follows, as we have agreed in class/Notes, we will *call* $\vdash (\exists \mathbf{x})A \equiv \neg(\forall \mathbf{x})\neg A$ "Definition of E".

NOTE that the 1-point rule proved in class (for ∀) under the exact same assumptions as stated in the 1st paragraph above is

$$\vdash (\forall \mathbf{x})(\mathbf{x} = t \wedge A) \equiv A[\mathbf{x} := t] \tag{1}$$

The proof of the E-version is below:

$$
\begin{aligned}
& (\exists \mathbf{x})(\mathbf{x} = t \wedge A) \\
\Leftrightarrow & \langle \text{Def. of E} \rangle \\
& \neg(\forall \mathbf{x})\neg(\mathbf{x} = t \wedge A) \\
\Leftrightarrow & \Big\langle \text{WL + Ax. 1 (hence abs. thm); Denom: } \neg(\forall \mathbf{x})\mathbf{p}^1 \Big\rangle \\
& \neg(\forall \mathbf{x})(\mathbf{x} = t \to \neg A) \\
\Leftrightarrow & \langle \text{WL + 1-point rule for } \forall \text{ (abs. thm!); Denom: } \neg \mathbf{p} \rangle \\
& \neg\Big(\neg A[\mathbf{x} := t\Big) \\
\Leftrightarrow & \langle \neg\neg\text{-thm} \rangle \\
& A[\mathbf{x} := t]
\end{aligned}
$$

□

[1] Using **Ax.1** as "$\neg(X \wedge Y) \equiv X \to \neg Y$" where "$X$" is "$\mathbf{x} = t$" and "$Y$" is "$A$".

Predicate Logic 3. (5 MARKS) You must use the technique of the "**auxiliary hypothesis metatheorem**" in the proof that you are asked to write here.

Any other proof (***even IF correct***) will *MAX* at 0 MARKS.

For any formulas A, B prove that

$$\vdash (\exists x)(A \wedge B) \to (\exists x)A \wedge (\exists x)B$$

Proof. By DThm prove instead

$$(\exists x)(A \wedge B) \vdash (\exists x)A \wedge (\exists x)B$$

1)	$(\exists x)(A \wedge B)$	$\langle$hyp$\rangle$
2)	$A[x := z] \wedge B[x := z]$	$\langle$aux. hyp for 1; z is fresh$\rangle$
3)	$A[x := z]$	$\langle$2 + Post$\rangle$
4)	$B[x := z]$	$\langle$2 + Post$\rangle$
5)	$(\exists x)A$	$\langle$3 + Dual Spec$\rangle$
6)	$(\exists x)B$	$\langle$4 + Dual Spec$\rangle$
7)	$(\exists x)A \wedge (\exists x)B$	$\langle$5 + 6 + Post$\rangle$

□

Predicate Logic 4. (5 MARKS) Use 1st-Order Soundness to prove that

$$\nvdash (\exists x)A \wedge (\exists x)B \rightarrow (\exists x)(A \wedge B) \tag{1}$$

that is, $(\exists x)A \wedge (\exists x)B \rightarrow (\exists x)(A \wedge B)$ **is NOT a theorem of predicate logic**.

Hint. Use a **countermodel** for a simple instant of the wff in (1), where you chose appropriate **atomic** A and B.

Proof. I interpret each of A and B as atomic formulas of <u>arithmetic</u>. Namely,

A stands for $x > 42$ and

B stands for $x < 42$.

So the interpretation of the wff in (1) over $\mathfrak{D} = (\mathbb{N}, M)$ is

$$\overbrace{(\exists x \in \mathbb{B})x > 42}^{\mathbf{t}} \wedge \overbrace{(\exists x \in \mathbb{N})x < 42}^{\mathbf{t}} \rightarrow \overbrace{(\exists x \in \mathbb{N})(x > 42 \wedge x < 42)}^{\mathbf{f}} \tag{2}$$

(2) being false (see **t**/**f** markings above) we have found a <u>countermodel</u> of a <u>special case</u> of the wff schema in (1). Said special case is NOT a theorem and thus the wff schema in (1) is not a theorem schema (because one of its <u>instances</u> —(2)— is NOT a theorem of arithmetic). □