York University Department of Electrical Engineering and Computer Science Lassonde School of Engineering

MATH 1090A. <u>MID TERM</u>, October 25, 2023; SOLUTIONS Professor George Tourlakis

Question 1. (3 MARKS) Prove that <u>NO wff</u> is the empty string λ by showing it contains AT LEAST ONE symbol from the Boolean alphabet.

The proof must be by analysing either formula constructions, or by the recursive definition of formulas.

Proof.

(a) By formula construction/calculation.

Well, a wff is a string that *appears* by itself on a <u>line</u> of a formula calculation. What do we legally write on such a line? <u>ONE</u> of the following:

- An atomic wff, such as \bot , \top , **p**. By inspection, none of these is λ .
- A string (¬B) licensed by the fact that we wrote the string B earlier. But this contains, for example ¬ —also "(" and ")"— so it too is not λ.
- A string (B ∘ C) —where ∘ ∈ {∧, ∨, →, ≡} <u>licensed</u> by the fact that we wrote the strings B and C earlier. But this string, this wff, contains, for example ∘ —but also "(" and ")"— so it too is not λ.

So, none of the strings I may <u>EVER</u> write in a step of a formula calculation can be λ . <u>THUS</u>, no wff can be λ as a wff is *precisely* a string *I may write* in a step of a formula calculation! In short, no wff can be λ

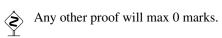
(b) By recursive formula definition.

Show that any wff A is $\neq \lambda$.

Well, A is one of the following three:

- \bot , \top , **p**. Each of them $\neq \lambda$.
- $(\neg B)$. NOT λ ! Contains, say, " \neg " (no quotes).
- $(B \circ C)$. NOT λ ! Contains, say, " \circ " (no quotes).

Question 2. (4 MARKS) Give an Equational proof of $\vdash A \rightarrow B \equiv A \land B \equiv A$.



Proof.

$$\begin{array}{l} A \rightarrow B \\ \Leftrightarrow \quad \langle \mathrm{axiom} \rangle \\ A \lor B \equiv B \\ \Leftrightarrow \quad \langle \mathrm{GR} \ \mathrm{axiom} \rangle \\ A \land B \equiv A \end{array}$$

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Question 3. (5 MARKS) Give an **Equational proof** of the following:

$$\vdash A \lor B \equiv (A \to \bot) \to B$$

Any other proof maxes to zero!

Proof.

$A \vee B$	
\Leftrightarrow	$\langle \text{Leib+double neg; Denom: } \mathbf{p} \lor B \rangle$
$\neg \neg A \lor B$	
\Leftrightarrow	$\langle \neg \lor$ -thm \rangle
$\neg A \rightarrow B$	
\Leftrightarrow	$\langle \text{Leib+thm; Denom: } \mathbf{p} \rightarrow B \rangle$
$(\neg A \lor \bot) \to B$	Comment : this "obvious" step I had omitted; I should get "3/5" in my old proof! :)
\Leftrightarrow	$\langle \text{Leib+}\neg \lor \text{-thm}; \text{Denom: } \mathbf{p} \rightarrow B \rangle$
$(A \to \bot) \to B$	

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Question 4. (a) (4 MARKS) For any A and B prove the following via an Equational Proof.

$$\vdash A \land \neg A \to B$$

Any other proof maxes to a zero.

Proof.

$$\begin{array}{l} A \wedge \neg A \rightarrow B \\ \Leftrightarrow \quad \langle \neg \lor \text{-thm} \rangle \\ \neg (A \wedge \neg A) \lor B \\ \Leftrightarrow \quad \langle \text{deMorgan + Leib; Denom: } \neg \mathbf{p} \lor B \rangle \\ \neg \neg (\neg A \lor A) \lor B \\ \Leftrightarrow \quad \langle \text{double neg + Leib; Denom: } \mathbf{p} \lor B \rangle \\ (\neg A \lor A) \lor B \\ \Leftrightarrow \quad \langle \text{Red. } \top \text{METAthm + axiom + Leib; Denom: } \mathbf{p} \lor B \rangle \\ \top \lor B & \text{bingo!} \end{array}$$

(b) (2 MARKS) Use a Hilbert style proof AND the above result to prove

 $A \wedge \neg A \vdash B$

Any other proof maxes to a zero. **Proof**.

(1)	$A \wedge \neg A$	$\langle hyp \rangle$
(2)	$A \wedge \neg A \to B$	\langle thm in subquestion (a) \rangle
(3)	B	$\langle 1 + 2 + modus \ ponens \ (MP) \rangle$

Extra blank "overflow" page for answers