York University

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MATH 2090. Problem Set #5. Posted March 17, 2002

Due in the Course Box by 2pm. April 5, 2002

Section N

1. Let A be a finite alphabet. As usual, A^* denotes the set of *all* strings formed using symbols in A.

   Let us define string reversal, x^R, that is, x^R is what you get if you read x backwards (right to left). Throughout, ε is the empty string and $*$ denotes concatenation.

   Formal definition:

   $$\begin{aligned} \varepsilon^R &= \varepsilon \\ (x * a)^R &= a * x^R, \text{ for all } a \in A \text{ and } x \in A^* \end{aligned}$$

   We say that x is a *substring* of y iff (for some u, v) $y = u*x*v$. Clearly, the substring relation has the finite descending chain condition, so that it admits (CVI) induction.

   Using the *formal* definition of x^R above and induction on the substring relation prove that for all strings x and y,

   $$(x * y)^R = y^R * x^R$$

2. Prove by induction on the subformula relation that if A is a syntactically well formed formula of propositional calculus, then every proper nonempty prefix of A has more left than right parentheses.

   Clarification: If x is any string, then z is a *prefix* of x iff there is a string y such that $x = z * y$, where "$*$" denotes concatenation. z is a *proper prefix* iff $z \neq x$ (equivalently, $y \neq \varepsilon$, i.e., the "tail" y is not the empty string).

3. Let $\rho : A \to A$ be given, and $A = \{1, 2, 3, \ldots, n\}$. Prove that

$$\rho^+ = \bigcup_{i=1}^{n} \rho^i$$

Hint. We know that $\rho^+ = \bigcup_{i \geq 1} \rho^i$. Thus you must prove only $\bigcup_{i \geq 1} \rho^i \subseteq \bigcup_{i=1}^{n} \rho^i$, the opposite inclusion being trivial. To this end prove

$$x \rho^k y \Rightarrow (\exists i \leq n) x \rho^i y \tag{1}$$

Obviously, the only case that needs work in (1) is when $k > n$. Well, by a cut-and-paste argument show that if $k > n$ and $x\rho^k y$, then there is also a $k' < k$ such that $x \rho^{k'} y$.

Two things: First, this helps, for you can't have an infinite sequence of "k" values

$$n < \ldots < k''' < k'' < k' < k$$

Second, recall that $x\rho^k y$ means (right?)

$$x \rho x_1 \rho x_2 \rho x_3 \cdots x_{k-1} \rho y, \text{ for some } x_i \in A, i = 1, \ldots, k-1$$

Can all the x_i be *different*? (It is good to split into two cases, $x = y$ and $x \neq y$.)

4. Let ρ and A be as above, but *moreover*, now we assume that ρ is reflexive on A. Prove that

$$\rho^+ = \rho^{n-1}$$

5. Let $f : A \to B$ be a partial function. Let X and Y be subsets of A, while U and V are subsets of B. Prove

(a) $f[X \cup Y] = f[X] \cup f[Y]$

(b) $f[X \cap Y] \subseteq f[X] \cap f[Y]$

(c) Can the "$\subseteq$" above be "sharpened" into an "$=$"? **WHY? (Do <u>not</u> explain in words if your answer is "no". Give a counterexample instead. If you said "yes", prove it.**

(d) $f^{-1}[U \cap V] = f^{-1}[U] \cap f^{-1}[V]$

"f^{-1}" above denotes *inverse relation.* No 1-1ness assumption is being made!