MA2441 3.0. Problem Set No. 5. (On Logic)

Dept. of Mathematics (Atkinson College)

Date: June 28, 1999
Due: July 12, 1999

Please note: The Final Take-Home Exam will be handed out in class, on Monday, July 12, 1999. **Don't miss it!**

It will be due (**in class—by 7:15pm**) on July 19, 1999 (**there will be absolutely no extensions**).

1. Prove by a *syntactic* argument that if $\vdash A \to B$, then $\vdash (\forall x)A \to (\forall x)B$ for any formulas A and B.

Proof.

(1) $A \to B$ (proved *without* nonlogical axioms—by assumption)

(2) $(\forall x)(A \to B)$ ((1) + generalization; *OK to do this since (1) was obtained w/o nonlog. axioms*)

(3) $(\forall x)(A \to B) \to \big((\forall x)A \to (\forall x)B\big)$ (proved w/o nonlog. axioms—from class)

(4) $(\forall x)A \to (\forall x)B$ ((2) + (3) + Modus Ponens) □

2. Using *resolution* prove the most general rule of "proof by cases", namely:

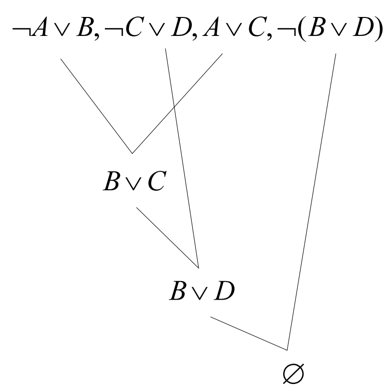
$$A \to B, C \to D \vdash A \vee C \to B \vee D$$

A proof-by-truth table will *not* be accepted in this exercise.

Proof. Using the Deduction Theorem, we need to show

$$A \to B, C \to D, A \vee C \vdash B \vee D$$

that is, find a contradiction in $\neg A \vee B, \neg C \vee D, A \vee C, \neg(B \vee D)$. Here it goes:



□

3. Prove by a *syntactic* argument that *if x is not free in B*, then

$$\vdash (\forall x)A \vee B \leftrightarrow (\forall x)(A \vee B) \qquad (i)$$

for any wff A and B.

Proof. ($\rightarrow$ direction): We split this into two "cases", namely "$(\forall x)A$" and "B":
Thus, we show first

$$\vdash (\forall x)A \rightarrow (\forall x)(A \vee B) \qquad (ii)$$

So,
(1) $A \rightarrow A \vee B$ (logical axiom—a tautology!)
(2) $(\forall x)A \rightarrow (\forall x)(A \vee B)$ (by (1) and problem 1). Done.

We also need

$$\vdash B \rightarrow (\forall x)(A \vee B) \qquad (iii)$$

So,
(3) $B \rightarrow A \vee B$ (logical axiom—a tautology!)
(4) $(\forall x)B \rightarrow (\forall x)(A \vee B)$ (by (3) and problem 1).

Now, by generalization, $B \vdash (\forall x)B$ since x is *not free* in B.†
By the DThm, again, $\vdash B \rightarrow (\forall x)B$, hence $\vdash B \leftrightarrow (\forall x)B$ by $\vdash (\forall x)B \rightarrow B$ (specialization).
This discussion, and the Leibniz rule, yield
(5) $B \rightarrow (\forall x)(A \vee B)$ from (4).

Equipped with (ii) and (iii) we conclude the proof of (i) [$\rightarrow$ direction] as follows:

By problem 2,

$$(\forall x)A \rightarrow (\forall x)(A\vee B), B \rightarrow (\forall x)(A\vee B) \vdash (\forall x)A\vee B \rightarrow (\forall x)(A\vee B)\vee(\forall x)(A\vee B) \qquad (6)$$

Since the premises in (6) have been proved *without nonlog. axioms*, so is the conclusion, and hence by Leibniz and the tautology $C \vee C \leftrightarrow C$, we have (i) ($\rightarrow$ direction).

($\leftarrow$ direction): By DThm, prove $(\forall x)(A \vee B) \vdash (\forall x)A \vee B$.
(a) $(\forall x)(A \vee B)$ (the "let" [hypothesis])
(b) $(\forall x)(\neg B \rightarrow A)$ ((a) + Leibniz rule)
(c) $(\forall x)\neg B \rightarrow (\forall x)A$ (from (b). See steps (2)–(4) in problem 1)
(d) $\neg B \rightarrow (\forall x)A$ ((c) + Leibniz; recall: $\neg B$ has no free x)
(e) $(\forall x)A \vee B$ ((d) + tautological implication). □

A simpler proof uses Leibniz on

$$\vdash (\forall x)(A \rightarrow B) \leftrightarrow (A \rightarrow (\forall x)B)$$

†The actual steps for that, (*required!*) are: $B \vdash B$ since (DThm) this is the same as $\vdash B \rightarrow B$ which is correct (tautology). Take now the set of premises, "S", to be $\{B\}$. Thus, from $S \vdash B$ and the assumption on x, $S \vdash (\forall x)B$.

where A has no free x.

There is a catch: We had not covered in class the $\rightarrow$ direction of the above, so if you went that way, you should have proved that direction yourselves!

4. *True* or *False*? Give reasons.

(a) A propositional variable *must* have an *intrinsic* true or false meaning. **FALSE. We can assign arbitrary values (true/false) to propositional variables.**

(b) Propositional variables can *only* denote mathematical formulas that contain no variables (e.g., $(\forall x)x = 4$ and $1 = 2 - 1$ are OK, but $x = 5$ and $x < y$ are not). **FALSE. Propositional variables can denote any *atomic* formula and any formula such as** $((\forall x)A)$ **or** $((\exists x)A)$**.**

(c) Propositional Calculus studies the properties of "propositions". **FALSE. It studies the properties of the connectives.**

5. Which of the following are "predicates"? Which are "atomic formulas"? *Explain. Use class-notes since the text is in error here!*

(a) $x \in y$ **ATOMIC (since "$\in$" is a predicate of arity 2, and** x, y **are terms).**

(b) $x = y$ **ATOMIC (since "$=$" is a predicate of arity 2, and** x, y **are terms).**

(c) $\in$ **A predicate (symbol) of set theory.**

(d) $=$ **A predicate (symbol) of Logic.**

(e) P, where $P(x)$ stands for "$x > 3$ & $(\exists y)x = y^2$" **A (defined) predicate (symbol).**

(f) $P(x)$ above. **ATOMIC (since "P" is a predicate of arity 1, and** x **is a term).**