MA3190.03

Problem Set No. 1—Solutions

Dept. of Mathematics and Statistics

1. Suppose we defined "$((\exists x)\mathcal{A}[x])$ is true" to mean that for some value of x in the relevant domain, say c, $\mathcal{A}[c]$ is true. Under such definition, is $(\exists x)y < x$ true over $\mathbb{R}$? How about under the "normal" definition?

Answer. (1) Under this definition, "$(\exists x)y < x$ true" means that "for some real c, $y < c$ is true". This does not hold, for the reals are unbounded above.
(2) Under the "normal" definition, "$(\exists x)y < x$ true" means "$(\forall y)(\exists x)y < x$ true", in words, "for each real y there is a larger real x". This is true.

2. Let a be a set, and consider the class $b = \{x \in a : x \notin x\}$.
Show that, despite similarities with the Russell class R, b *is* a set.
Moreover, show that $b \notin a$.

Answer. (1) $b \subseteq a$, hence b is a set by separation.
(2) Suppose $b \in a$. Then $b \in a \;\&\; b \notin b$ is true (since $b \notin b$ is true by foundation). Thus, $b \in b$, which is a contradiction (foundation again). We conclude that $b \notin a$.

Here is a better proof. "Better" because it is more "general", since it does not use foundation: We have

$$x \in b \leftrightarrow x \in a \;\&\; x \notin x$$

thus

$$b \in b \leftrightarrow b \in a \;\&\; b \notin b$$

Now assume that $b \in a$ is true. The above then yields the contradiction†

$$b \in b \leftrightarrow b \notin b$$

3. Show that R (the Russell class)$= \mathbb{U}$.

Answer. By the way, by "$\mathbb{U}$" we mean $\mathbb{U}_M$.

We have shown in class that $R \subseteq \mathbb{U}$ (from which, as you recall, we inferred that $\mathbb{U}$ is a proper class).

We only need $\supseteq$. This follows from $x = x \to x \notin x$, since $x \notin x$ is true by foundation.

† Thinking semantically, we have got rid of "$b \in a$" by applying $\mathcal{A} \leftrightarrow \mathbf{t} \;\&\; \mathcal{A}$.
Alternatively, thinking formally, we have applied $\mathcal{A} \vdash \mathcal{B} \leftrightarrow \mathcal{A} \;\&\; \mathcal{B}$.

4. Show that if a class $\mathbb{A}$ satisfies $\mathbb{A} \subseteq \mathbb{X}$ for all $\mathbb{X}$, then $\mathbb{A} = \emptyset$.

Answer. Taking $\mathbb{X} = \emptyset$ we get $\mathbb{A} \subseteq \emptyset$. Since $x \neq x \rightarrow x \in \mathbb{A}$ is true, we also have $\emptyset \subseteq \mathbb{A}$, hence $\mathbb{A} = \emptyset$ by the definition of equality for class-terms.

5. Without using foundation, show that $\emptyset \neq \{\emptyset\}$.

Answer. If $\emptyset = \{\emptyset\}$ then

$$x \in \emptyset \leftrightarrow x \in \{\emptyset\}$$

hence

$$\emptyset \in \emptyset \leftrightarrow \emptyset \in \{\emptyset\}$$

which is false (the left of $\leftrightarrow$ is false, the right is true).

6. What is $\bigcap \emptyset$ (and why)?

Answer.

$$\begin{aligned} \bigcap \emptyset &= \{x : (\forall y)(y \in \emptyset \rightarrow x \in y)\} \\ &= \mathbb{U}_M \end{aligned}$$

since $(\forall y)(y \in \emptyset \rightarrow x \in y)$ is true (provable) for all x.

7. For any set A, show that $\mathbb{U} - A$ is a proper class.

Answer. If not, let $\mathbb{U} - A = B$, a set. Then $\mathbb{U} = A \cup B$, a set.

8. Show for any classes $\mathbb{A}, \mathbb{B}$, that $\mathbb{A} - \mathbb{B} = \mathbb{A} - \mathbb{A} \cap \mathbb{B}$.

Answer. $(\subseteq)$. Let $x \in \mathbb{A} - \mathbb{B}$. Then $x \in \mathbb{A}$ *and* $x \notin \mathbb{B}$. By definition of "$\cap$", $x \notin \mathbb{A} \cap \mathbb{B}$, as well. Thus, $x \in \mathbb{A} - \mathbb{A} \cap \mathbb{B}$.

$(\supseteq)$. Let $x \in \mathbb{A} - \mathbb{A} \cap \mathbb{B}$. Then $x \in \mathbb{A}$ *and* $x \notin \mathbb{A} \cap \mathbb{B}$. By definition of "$\cap$", $x \notin \mathbb{B}$, as well (since $x \in \mathbb{A}$). Thus, $x \in \mathbb{A} - \mathbb{B}$.

9. For any classes $\mathbb{A}, \mathbb{B}$ show that $\mathbb{A} - (\mathbb{A} - \mathbb{B}) = \mathbb{B}$ iff $\mathbb{B} \subseteq \mathbb{A}$.

Answer. *only if*-part. Let $x \in \mathbb{B}$. Then $x \in \mathbb{A} - (\mathbb{A} - \mathbb{B})$, hence $x \in \mathbb{A}$.

if-part. $(\subseteq)$. Let $x \in \mathbb{A} - (\mathbb{A} - \mathbb{B})$. Then $x \in \mathbb{A}$ *and* $x \notin \mathbb{A} - \mathbb{B}$, thus, $x \in \mathbb{B}$ (**NB.** The hypothesis, $\mathbb{B} \subseteq \mathbb{A}$, was not needed in this direction).

$(\supseteq)$. Let $x \in \mathbb{B}$. By hypothesis, $x \in \mathbb{A}$. Hence $x \notin \mathbb{A} - \mathbb{B}$. Thus, $x \in \mathbb{A} - (\mathbb{A} - \mathbb{B})$.

10. (1) Express $\mathbb{A} \cap \mathbb{B}$ using class difference as the only operation.

Answer. $\mathbb{A} \cap \mathbb{B} = \mathbb{A} - (\mathbb{A} - \mathbb{B})$. One can verify this by the traditional "let $x \in$ lhs . . . " and "let $x \in$ rhs . . . " arguments.

Here is an alternative, for variety's sake:

Let the classes $\mathbb{A}$ and $\mathbb{B}$ be given by the class-terms $\{x : \mathcal{A}(x)\}$ and $\{x : \mathcal{B}(x)\}$ respectively. By definition of equality between class-terms, one needs to verify that

$$\mathcal{A}(x) \ \& \ \mathcal{B}(x) \leftrightarrow \mathcal{A}(x) \ \& \ \neg(\mathcal{A}(x) \ \& \ \neg\mathcal{B}(x))$$

which follows at once by deMorgan's and distributive laws of logic.

(2) Express $\mathbb{A} \cup \mathbb{B}$ using class difference/complement as the only operations.

Answer. By deMorgan's laws,

$$\begin{aligned} \mathbb{A} \cup \mathbb{B} &= \overline{\overline{\mathbb{A}} \cap \overline{\mathbb{B}}} \\ &= \overline{\overline{\mathbb{A}} - (\overline{\mathbb{A}} - \overline{\mathbb{B}})} \end{aligned}$$

where we wrote $\overline{\mathbb{A}}$ for $\mathbb{U}_M - \mathbb{A}$.

11. Show that we cannot have $a \in b \in c \in \cdots \in a$.

Answer. To fix ideas, let us write $a_1, a_2, \ldots, a_n$ instead of the "$a, b, c, \ldots$" in the above "cycle".

So we are asked to prove that it is impossible to have

$$a_1 \in a_2 \in a_3 \in \ldots a_n \in a_1 \tag{1}$$

Let $A = \{a_1, \ldots, a_n\}$ (that is, $A = \{x : x = a_1 \vee x = a_1 \vee \ldots \vee x = a_n\}$).

NB. While A is a set,† we do not need this fact.

We proceed by contradiction, *assuming* (1).

By *foundation*, let $x \in A$ be such that $\neg(\exists y \in A) y \in x$.

By (1) such an x does not exist, for if it does, it is some a_j. But then $a_m \in a_j$, where $m = n$ if $j = 1$, $m = j - 1$ otherwise.

† This is shown by *metamathematical* induction on the informal "index", n, using pairing and union. The induction step shows that if $\{a_1, \ldots, a_n\}$ is a set then so is $\{a_1, \ldots, a_n, a_{n+1}\} = \bigcup\{\{a_1, \ldots, a_n\}, \{a_{n+1}\}\}$.