MA3190.03

Problem Set No. 2—Solutions

Dept. of Mathematics and Statistics

1. Show that for any class (not just set) $\mathbb{A}$, $\mathbb{A} \in \mathbb{A}$ is false.

Answer. $\mathbb{A} \in \mathbb{A}$ is true, then $\mathbb{A}$ is a set (for class terms are collections of sets or atoms, and $\mathbb{A}$ is a member of a class term: $\mathbb{A}$). But that is absurd by foundation.

2. (1) Show that $\mathbb{A} =$"the class of *all* sets that contain at least one element" can be defined by a class-term.
(2) Show that $\mathbb{A}$ is a proper class.

Answer. (1) If $\mathbb{A} = \{x : \neg\mathcal{U}(x) \ \ \& \ \ (\exists y) y \in x\}$.

(2) Consider the class $\mathbb{B} = \{\{x\} : x = x\}$. By replacement ($\{x\} \mapsto x$), $\mathbb{B}$ is not a set (else so would be $\mathbb{U}_M$).

Since $\mathbb{B} \subseteq \mathbb{A}$, $\mathbb{A}$ is a proper class by separation.

3. Attach the intuitive meaning to the statement that the set A has n (distinct) elements.

Show then by induction on n, that for $n \geq 0$, if A has n elements, then $\mathbf{P}(A)$ has 2^n elements.

Answer. Let $A = \{a_1, \ldots, a_n\}$. As it is usual, $n = 0$ means $A = \emptyset$.
So let $n = 0$. Then $\mathbf{P}(A) = \{\emptyset\}$, so it has exactly $1 = 2^0$ element.
Assume the claim for n (fixed) and go to $n + 1$.
Now $B = A \cup \{a_{n+1}\}$ is the set we start with (it has $n + 1$ elements).
Clearly, $\mathbf{P}(B) = \mathbf{P}(A) \cup \{x \cup \{a_{n+1}\} : x \in \mathbf{P}(A)\}$.
Thus, $\mathbf{P}(B)$ has exactly twice as many members as $\mathbf{P}(A)$, that is (using I.H.), $2^n + 2^n = 2^{n+1}$.

4. Show (without the use of foundation) that $\big\{\{a\}, \{a, b\}\big\} = \big\{\{a'\}, \{a', b'\}\big\}$ implies $a = a'$ and $b = b'$.

Answer. $\bigcap \big\{\{a\}, \{a, b\}\big\} = \bigcap \big\{\{a'\}, \{a', b'\}\big\}$, that is, $\{a\} = \{a'\}$, hence

$$a = a' \tag{1}$$

Similarly, $\bigcup\{\{a\},\{a,b\}\} = \bigcup\{\{a'\},\{a',b'\}\}$, that is, by (1),

$$\{a,b\} = \{a,b'\}$$

From this last equality, exactly as in the text, it follows that $b = b'$ (recall that we did not use foundation in the text for this last implication).

5. For any sets x, y show that $x \cup \{x\} = y \cup \{y\} \to x = y$.
(*Hint*: Use foundation.)

Answer. In the text.

6. For any $\mathbb{A}, \mathbb{B}$ show that $\emptyset = \mathbb{A} \times \mathbb{B}$ iff $\mathbb{A} = \emptyset$ or $\mathbb{B} = \emptyset$.

Answer. $(\to)$ Do the contrapositive: Let $\emptyset \neq \mathbb{A}$ and $\emptyset \neq \mathbb{B}$. Then, let $a \in \mathbb{A}$ and $b \in \mathbb{B}$. Therefore, $\langle a,b\rangle \in \mathbb{A} \times \mathbb{B}$. Hence $\emptyset \neq \mathbb{A} \times \mathbb{B}$.

$(\leftarrow)$ Suppose $\emptyset = \mathbb{A}$. Now, $\langle x,y\rangle \in \mathbb{A} \times \mathbb{B}$ iff $x \in \mathbb{A}$ & $y \in \mathbb{B}$. But the rhs of "iff" is false, for "$x \in \mathbb{A}$" is.

7. Show that $\mathbb{U}^3_M \subseteq \mathbb{U}^2_M$.

Answer. Let $\langle x,y,z\rangle \in \mathbb{U}^3_M$. But $\langle x,y,z\rangle = \langle\langle x,y\rangle, z\rangle \in \mathbb{U}^2_M$.

8. Let $\mathbb{F} : \mathbb{X} \to \mathbb{Y}$ be a function, and $\mathbb{A} \subseteq \mathbb{Y}$, $\mathbb{B} \subseteq \mathbb{Y}$. Prove
(a) $\mathbb{F}^{-1}[\mathbb{A} \cup \mathbb{B}] = \mathbb{F}^{-1}[\mathbb{A}] \cup \mathbb{F}^{-1}[\mathbb{B}]$
(b) $\mathbb{F}^{-1}[\mathbb{A} \cap \mathbb{B}] = \mathbb{F}^{-1}[\mathbb{A}] \cap \mathbb{F}^{-1}[\mathbb{B}]$
(c) if $\mathbb{A} \subseteq \mathbb{B}$, then $\mathbb{F}^{-1}[\mathbb{B} - \mathbb{A}] = \mathbb{F}^{-1}[\mathbb{B}] - \mathbb{F}^{-1}[\mathbb{A}]$.
Is this last equality true if $\mathbb{A} \not\subseteq \mathbb{B}$? Why?

Answer.
(a)

$$\begin{aligned} x \in \mathbb{F}^{-1}[\mathbb{A} \cup \mathbb{B}] &\leftrightarrow \mathbb{F}(x) \in \mathbb{A} \cup \mathbb{B} \\ &\leftrightarrow \mathbb{F}(x) \in \mathbb{A} \vee \mathbb{F}(x) \in \mathbb{B} \\ &\leftrightarrow x \in \mathbb{F}^{-1}[\mathbb{A}] \vee x \in \mathbb{F}^{-1}[\mathbb{B}] \\ &\leftrightarrow x \in \mathbb{F}^{-1}[\mathbb{A}] \cup \mathbb{F}^{-1}[\mathbb{B}] \end{aligned}$$

(b)

$$\begin{aligned} x \in \mathbb{F}^{-1}[\mathbb{A} \cap \mathbb{B}] &\leftrightarrow \mathbb{F}(x) \in \mathbb{A} \cap \mathbb{B} \\ &\leftrightarrow \mathbb{F}(x) \in \mathbb{A} \ \& \ \mathbb{F}(x) \in \mathbb{B} \\ &\leftrightarrow x \in \mathbb{F}^{-1}[\mathbb{A}] \ \& \ x \in \mathbb{F}^{-1}[\mathbb{B}] \\ &\leftrightarrow x \in \mathbb{F}^{-1}[\mathbb{A}] \cap \mathbb{F}^{-1}[\mathbb{B}] \end{aligned}$$

(c)

$$\begin{aligned} x \in \mathbb{F}^{-1}[\mathbb{B} - \mathbb{A}] &\leftrightarrow \mathbb{F}(x) \in \mathbb{B} - \mathbb{A} \\ &\leftrightarrow \mathbb{F}(x) \in \mathbb{B} \ \& \ \mathbb{F}(x) \notin \mathbb{A} \\ &\leftrightarrow x \in \mathbb{F}^{-1}[\mathbb{B}] \ \& \ x \notin \mathbb{F}^{-1}[\mathbb{A}] \\ &\leftrightarrow x \in \mathbb{F}^{-1}[\mathbb{B}] - \mathbb{F}^{-1}[\mathbb{A}] \end{aligned}$$

Is this last equality true if $\mathbb{A} \not\subseteq \mathbb{B}$? Why? Yes, we did not need the assumption in getting (c).

9. Using only the axioms of union and separation, show that if a function $\mathbb{F}$ is a set, then so are both $\mathrm{dom}(\mathbb{F})$ and $\mathrm{ran}(\mathbb{F})$.

Answer. $\mathbb{F}$ is a class (here set) of pairs $\langle x, y\rangle$. Recall that $\langle x, y\rangle = \{x, \{x, y\}\}$. Thus,

$$\mathrm{dom}(\mathbb{F}) \subseteq \bigcup \mathbb{F}$$

and

$$\mathrm{ran}(\mathbb{F}) \subseteq \bigcup \bigcup \mathbb{F}$$

10. Show that if for a relation $\mathbb{S}$, both the range and the domain are sets, then $\mathbb{S}$ is a set.

Answer. $\mathbb{S} \subseteq \mathrm{dom}(\mathbb{S}) \times \mathrm{ran}(\mathbb{S})$.

11. Show for any relation S, that if S is a set then so is S^{-1}.

Answer. Directly, by replacement $\langle x, y\rangle \mapsto \langle y, x\rangle$.

Or, with more fuss, we know (by 9) that $\mathrm{dom}(S)$ and $\mathrm{ran}(S)$ are sets. $S^{-1} \subseteq \mathrm{ran}(S) \times \mathrm{dom}(S)$.