MA3190.03

Problem Set No. 3—Solutions

Dept. of Mathematics and Statistics

1. Prove that for any formula $\mathcal{F}(x)$,

$$(\forall n \in \omega)\big((\forall m < n \in \omega)\mathcal{F}(m) \to \mathcal{F}(n)\big) \vdash (\forall n \in \omega)\mathcal{F}(n)$$

or, in words, "if for any $n \in \omega$ we can prove $\mathcal{F}(n)$ on the *induction hypothesis* that $\mathcal{F}(m)$ holds for *all* $m < n$, then this is as good as having proved $(\forall n \in \omega)\mathcal{F}(n)$".

This type of induction is called *course-of-values induction*.

(*Hint.* Consider the formula $\mathcal{G}(n)$ defined as $(\forall m < n \in \omega)\mathcal{F}(m)$ and apply (ordinary) induction on n to prove—under the I.H. for $\mathcal{F}(x)$—that $(\forall n \in \omega)\mathcal{G}(n)$. Note how the "basis" is buried inside the I.H. of course-of-values induction.)

Answer. We follow the hint.
Basis. $\mathcal{G}(0)$ is $(\forall m)(m < 0 \to \mathcal{F}(m))$, which is true since $m < 0$ is false.

Assume $\mathcal{G}(n)$ for fixed n (I.H.) and proceed to prove $\mathcal{G}(n+1)$.
Now

$$\begin{aligned}
\mathcal{G}(n+1) &\equiv (\forall m)(m < n+1 \to \mathcal{F}(m)) \\
&\equiv (\forall m)(m < n \lor m = n \to \mathcal{F}(m)) \\
&\equiv (\forall m)(m < n \to \mathcal{F}(m) \ \ \& \ \ m = n \to \mathcal{F}(m)), && \text{by Logic} \\
&\equiv (\forall m)(m < n \to \mathcal{F}(m)) \ \ \& \ \ (\forall m)(m = n \to \mathcal{F}(m)), && \text{by more Logic} \\
&\equiv \mathcal{G}(n) \ \ \& \ \ (\forall m)(m = n \to \mathcal{F}(m)) \\
&\equiv \mathcal{G}(n) \ \ \& \ \ \mathcal{F}(n), && \text{by a bit more Logic}
\end{aligned}$$

Now, we have $\mathcal{G}(n)$ by I.H., hence we have $\mathcal{F}(n)$ since $(\forall m < n)\mathcal{F}(m) \to \mathcal{F}(n)$, i.e.,

$$\mathcal{G}(n) \to \mathcal{F}(n) \tag{1}$$

By the above equivalences, we got $\mathcal{G}(n+1)$, thus, by simple induction, we now have $\mathcal{G}(n)$. By (1), we also have $\mathcal{F}(n)$.

2. (*The "least" number principle over ω.*) Prove that every $\emptyset \neq A \subseteq \omega$ has a *minimal* element, i.e., an $n \in A$ such that for *no* $m \in A$ is it possible to have $m < n$. Do so *without* foundation, using instead course-of-values induction.

Answer. We argue by contradiction. Let $\emptyset \neq A \subseteq \omega$, yet A has no minimal elements. We contradict this by showing $\omega - A = \omega$ (what does this contradict?)

We use course-of-values induction:

Basis. $0 \in \omega - A$, otherwise $0 \in A$, and clearly then 0 would be a minimal element of A.

Assume the claim, that $m \in \omega - A$, for all $m \leq n$ (fixed n).

We now argue the case for $n+1$: Suppose $n+1 \notin \omega - A$. Then $n+1 \in A$. But then, $n+1$ is minimal in A, for no $m < n+1$ is in A, by I.H. Once more, we arrived at a (final) contradiction.

3. Redo the proof of Theorem 1.20 (existence part) so that it goes through *even if* trichotomy of $\in$ over ω did not hold.

Answer. we need to redo the passage (from the proof of 1.20) below:

"... hence, by collection, $\mathcal{F}$ is a set. So is then

$$\widehat{f} \stackrel{\text{def}}{=} \bigcup \mathcal{F} \tag{2}$$

Observe first that $\widehat{f}$ is a function: Let $\langle a, b\rangle \in \widehat{f}$ and also $\langle a, c\rangle \in \widehat{f}$. Then, by (2), $f(a) = b$ and $f'(a) = c$ for some f, f' in $\mathcal{F}$. Without loss of generality, applying *trichotomy*, $\mathrm{dom}(f) \in \mathrm{dom}(f')$† By uniqueness, $f = f' \restriction \mathrm{dom}(f)$ since both sides of $=$ satisfy the same recurrence on $\mathrm{dom}(f)$. "

OK, so, let $\langle a, b\rangle \in \widehat{f}$ and also $\langle a, c\rangle \in \widehat{f}$. Then, by (2), $f(a) = b$ and $f'(a) = c$ for some f, f' in $\mathcal{F}$. Let $n = \mathrm{dom}(f)$ while $m = \mathrm{dom}(f')$. Thus $a \in n \cap m$. But then $a+1 \subseteq n \cap m$ (Why?). By the uniqueness part of the proof, $f \restriction (a+1) = f' \restriction (a+1)$, in particular, $f(a) = f'(a)$.

4. Prove that a set x is a natural number iff it satisfies (1) and (2) below.
(1) it and all its members are transitive
(2) it and all its members are successors or $\emptyset$.

Answer. The *only if* is from the text (which theorems?).
Here is the *if*. We prove that if (1) and (2) are satisfied, then $x \in \omega$. Well, suppose not, and let x_0 be $\in$-minimal (by foundation) that satisfies (1) and (2), yet

$$x_0 \notin \omega \tag{3}$$

Let $x \in x_0$. Now, x is transitive (by (1)). Let $y \in x$. Then $y \in x_0$, again by (1). One more invocation of (1), yields that y is transitive.

† If $\mathrm{dom}(f) = \mathrm{dom}(f')$ then $f = f'$ by uniqueness, hence $b = c$.

Thus x satisfies (1).

x satisfies (2) as well: Indeed, by $x \in x_0$, x is a successor or 0 (since x_0 satisfies (2)). But if $y \in x$, then $y \in x_0$, so, a member of x is also a successor or 0.

Thus, x satisfies (1) and (2), hence, by $\in$-minimality of x_0, $x \in \omega$.

By (3), $x_0 \neq 0$. By (2), it is a successor. Say, $x_0 = z \cup \{z\}$. Since $z \in x_0$, we have just seen that $z \in \omega$. Hence $x_0 \in \omega$, since ω is inductive. We have just contradicted (3).