MA3190.03

Problem Set No. 3

Dept. of Mathematics and Statistics

Date: March 4, 1999
Due: In two weeks

1. Prove that for any formula $\mathcal{F}(x)$,

$$(\forall n \in \omega)\big((\forall m < n \in \omega)\mathcal{F}(m) \to \mathcal{F}(n)\big) \vdash (\forall n \in \omega)\mathcal{F}(n)$$

or, in words, "if for any $n \in \omega$ we can prove $\mathcal{F}(n)$ on the *induction hypothesis* that $\mathcal{F}(m)$ holds for *all* $m < n$, then this is as good as having proved $(\forall n \in \omega)\mathcal{F}(n)$".

This type of induction is called *course-of-values induction*.

(*Hint.* Consider the formula $\mathcal{G}(n)$ defined as $(\forall m < n \in \omega)\mathcal{F}(m)$ and apply (ordinary) induction on n to prove—under the I.H. for $\mathcal{F}(x)$—that $(\forall n \in \omega)\mathcal{G}(n)$. Note how the "basis" is buried inside the I.H. of course-of-values induction.)

2. (*The "least" number principle over* ω*.*) Prove that every $\emptyset \neq A \subseteq \omega$ has a *minimal* element, i.e., an $n \in A$ such that for *no* $m \in A$ is it possible to have $m < n$. Do so *without* foundation, using instead course-of-values induction.

3. Redo the proof of Theorem 1.20 (existence part) so that it goes through *even if* trichotomy of $\in$ over ω did not hold.

4. Prove that a set x is a natural number iff it satisfies (1) and (2) below.
(1) it and all its members are transitive
(2) it and all its members are successors or $\emptyset$.