MATH 6030.03

A/W 1999–2000

Date: October 19, 1999
Due: Soon

Problem Set No. 2

1) Let f be continuous on $[a, b]$ and $f(a) < c < f(b)$ for some *real* c.

Using "nonstandard" techniques, show that there is a *real* $d \in [a, b]$ such that $f(d) = c$.

2) (*Skolem's "paradox"*) We assume (as all reasonable people do) that the Set Theory based on the well known Zermelo-Fraenkel axioms (with the axiom of choice) *is* consistent. We normally call this theory "ZFC".

Since its language is at most enumerable (indeed finite: it only contains the predicate "$\in$") ZFC has some *enumerable* model, $\mathfrak{M} = \{M, \ldots\}$.

Thus, all sets of the theory have at most enumerable interpretations as subsets of M. Doesn't this contradict the fact that non-enumerable sets exist? Why not?

3) (a) Find in the literature, and list the Peano axioms and schemata for Arithmetic over the language $L = \{0, <, ', +, \times\}$, where x' denotes $x + 1$.

(b) Show that this theory has nonstandard models, i.e., models that are not isomorphic to the "standard model" $\mathfrak{N} = (N; 0, <, ', +, \times)$.

4) (This is a bit "open-ended") Present a complete proof in the metatheory (by induction) that any two models $\mathfrak{M} = (M, \ldots)$ and $\mathfrak{K} = (K, \ldots)$ of Peano arithmetic *are* isomorphic. Start your isomorphism by making the least elements of M and K correspond. Then, assuming that you have corresponded $m \in M$ with $k \in K$ take the smallest unused element in M (that is the "successor" of m in M) and correspond it with the smallest unused element in K (that is the "successor" of k in K). Fill in the details (recall that isomorphism is more than just a 1-1 correspondence).

Wait a minute! Didn't you prove in 3) above that *not all* models of Peano arithmetic are isomorphic? What's going on?