

FIRST-ORDER LOGIC

CHAPTER 8

Outline

- ◇ Why FOL?
- ◇ Syntax and semantics of FOL
- ◇ Fun with sentences
- ◇ Wumpus world in FOL

Pros and cons of propositional logic

Propositional logic is **declarative**: pieces of syntax correspond

Propositional logic allows partial/disjunctive/negated information
(unlike most data structures and databases)

Propositional logic is **compositional**:

meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and $P_{1,2}$

Meaning in propositional logic is **context-independent**
(unlike natural language, where meaning depends on context)

Propositional logic has very limited expressive power
(unlike natural language)

E.g., cannot say “pits cause breezes in adjacent squares”
except by writing one sentence for each square

First-order logic

Whereas propositional logic assumes world contains facts,
first-order logic (like natural language) assumes the world contains

- Objects: people, houses, numbers, theories, Ronald McDonald, baseball games, wars, centuries . . .
- Relations: red, round, bogus, prime, multistoried . . . ,
brother of, bigger than, inside, part of, has color, occurred at,
comes between, . . .
- Functions: father of, best friend, third inning of, one more than
. . .

Logics in general

| Language | Ontological Commitment | Epistemological Commitment |
|---------------------|----------------------------------|-------------------------------|
| Propositional logic | facts | true/false, |
| First-order logic | facts, objects, relations | true/false, |
| Temporal logic | facts, objects, relations, times | true/false, |
| Probability theory | facts | degree of |
| Fuzzy logic | facts + degree of truth | known int |

Syntax of FOL: Basic elements

Constants *KingJohn, 2, UCB, ...*

Predicates *Brother, >, ...*

Functions *Sqrt, LeftLegOf, ...*

Variables *x, y, a, b, ...*

Connectives $\wedge \vee \neg \Rightarrow \Leftrightarrow$

Equality $=$

Quantifiers $\forall \exists$

Atomic sentences

Atomic sentence = $\textit{predicate}(\textit{term}_1, \dots, \textit{term}_n)$
or $\textit{term}_1 = \textit{term}_2$

Term = $\textit{function}(\textit{term}_1, \dots, \textit{term}_n)$
or *constant* or *variable*

E.g., $\textit{Brother}(\textit{KingJohn}, \textit{RichardTheLionheart})$
> $(\textit{Length}(\textit{LeftLegOf}(\textit{Richard})), \textit{Length}(\textit{LeftLegOf}(\textit{KingJohn})))$

Complex sentences

Complex sentences are made from atomic sentences using connectives

$$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \Rightarrow S_2, \quad S_1 \Leftrightarrow S_2$$

E.g. $Sibling(KingJohn, Richard) \Rightarrow Sibling(Richard, KingJohn)$

$$>(1, 2) \vee \leq(1, 2)$$

$$>(1, 2) \wedge \neg >(1, 2)$$

Truth in first-order logic

Sentences are true with respect to a model and an interpretation

Model contains ≥ 1 objects (domain elements) and relations am

Interpretation specifies referents for

constant symbols \rightarrow objects

predicate symbols \rightarrow relations

function symbols \rightarrow functional relations

An atomic sentence $predicate(term_1, \dots, term_n)$ is true
iff the objects referred to by $term_1, \dots, term_n$
are in the relation referred to by $predicate$

Models for FOL: Example

Truth example

Consider the interpretation in which

Richard \rightarrow Richard the Lionheart

John \rightarrow the evil King John

Brother \rightarrow the brotherhood relation

Under this interpretation, *Brother*(*Richard*, *John*) is true just in case Richard the Lionheart and the evil King John are in the brotherhood relation in the model

Models for FOL: Lots!

Entailment in propositional logic can be computed by enumeration

We can enumerate the FOL models for a given KB vocabulary:

For each number of domain elements n from 1 to ∞

For each k -ary predicate P_k in the vocabulary

For each possible k -ary relation on n objects

For each constant symbol C in the vocabulary

For each choice of referent for C from n objects .

Computing entailment by enumerating FOL models is not easy!

Universal quantification

$\forall \langle variables \rangle \langle sentence \rangle$

Everyone at Berkeley is smart:

$\forall x \text{ } At(x, Berkeley) \Rightarrow Smart(x)$

$\forall x \text{ } P$ is true in a model m iff P is true with x being each possible object in the model

Roughly speaking, equivalent to the conjunction of instantiations

$$\begin{aligned} & (At(KingJohn, Berkeley) \Rightarrow Smart(KingJohn)) \\ \wedge & (At(Richard, Berkeley) \Rightarrow Smart(Richard)) \\ \wedge & (At(Berkeley, Berkeley) \Rightarrow Smart(Berkeley)) \\ \wedge & \dots \end{aligned}$$

A common mistake to avoid

Typically, \Rightarrow is the main connective with \forall

Common mistake: using \wedge as the main connective with \forall :

$$\forall x \text{ } At(x, Berkeley) \wedge Smart(x)$$

means “Everyone is at Berkeley and everyone is smart”

Existential quantification

$\exists \langle variables \rangle \langle sentence \rangle$

Someone at Stanford is smart:

$\exists x \text{ } At(x, Stanford) \wedge Smart(x)$

$\exists x \text{ } P$ is true in a model m iff P is true with x being
some possible object in the model

Roughly speaking, equivalent to the disjunction of instantiations

$$\begin{aligned} & (At(KingJohn, Stanford) \wedge Smart(KingJohn)) \\ \vee & (At(Richard, Stanford) \wedge Smart(Richard)) \\ \vee & (At(Stanford, Stanford) \wedge Smart(Stanford)) \\ \vee & \dots \end{aligned}$$

Another common mistake to avoid

Typically, \wedge is the main connective with \exists

Common mistake: using \Rightarrow as the main connective with \exists :

$$\exists x \text{ } At(x, Stanford) \Rightarrow Smart(x)$$

is true if there is anyone who is not at Stanford!

Properties of quantifiers

$\forall x \forall y$ is the same as $\forall y \forall x$ (why??)

$\exists x \exists y$ is the same as $\exists y \exists x$ (why??)

$\exists x \forall y$ is **not** the same as $\forall y \exists x$

$\exists x \forall y \text{ Loves}(x, y)$

“There is a person who loves everyone in the world”

$\forall y \exists x \text{ Loves}(x, y)$

“Everyone in the world is loved by at least one person”

Quantifier duality: each can be expressed using the other

$\forall x \text{ Likes}(x, \text{IceCream}) \quad \neg \exists x \neg \text{Likes}(x, \text{IceCream})$

$\exists x \text{ Likes}(x, \text{Broccoli}) \quad \neg \forall x \neg \text{Likes}(x, \text{Broccoli})$

Fun with sentences

Brothers are siblings

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$$\forall x, y \text{ } Brother(x, y) \Rightarrow Sibling(x, y).$$

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“Sibling” is symmetric

$$\forall x, y \text{ } Sibling(x, y) \Leftrightarrow Sibling(y, x).$$

One's mother is one's female parent

Fun with sentences

Brothers are siblings

$$\forall x, y \text{ } Brother(x, y) \Rightarrow Sibling(x, y).$$

“Sibling” is symmetric

$$\forall x, y \text{ } Sibling(x, y) \Leftrightarrow Sibling(y, x).$$

One's mother is one's female parent

$$\forall x, y \text{ } Mother(x, y) \Leftrightarrow (Female(x) \wedge Parent(x, y)).$$

A first cousin is a child of a parent's sibling

Fun with sentences

Brothers are siblings

$$\forall x, y \text{ Brother}(x, y) \Rightarrow \text{Sibling}(x, y).$$

“Sibling” is symmetric

$$\forall x, y \text{ Sibling}(x, y) \Leftrightarrow \text{Sibling}(y, x).$$

One's mother is one's female parent

$$\forall x, y \text{ Mother}(x, y) \Leftrightarrow (\text{Female}(x) \wedge \text{Parent}(x, y)).$$

A first cousin is a child of a parent's sibling

$$\forall x, y \text{ FirstCousin}(x, y) \Leftrightarrow \exists p, ps \text{ Parent}(p, x) \wedge \text{Sibling}(\text{Parent}(ps, y), p)$$

Equality

$term_1 = term_2$ is true under a given interpretation
if and only if $term_1$ and $term_2$ refer to the same object

E.g., $1 = 2$ and $\forall x \times(Sqrt(x), Sqrt(x)) = x$ are satisfiable
 $2 = 2$ is valid

E.g., definition of (full) *Sibling* in terms of *Parent*:

$$\forall x, y \text{ Sibling}(x, y) \Leftrightarrow [\neg(x = y) \wedge \exists m, f \neg(m = f) \wedge \\ \text{Parent}(m, x) \wedge \text{Parent}(f, x) \wedge \text{Parent}(m, y) \wedge \text{Parent}(f, y)]$$

Interacting with FOL KBs

Suppose a wumpus-world agent is using an FOL KB and perceives a smell and a breeze (but no glitter) at $t = 5$:

$Tell(KB, Percept([Smell, Breeze, None], 5))$

$Ask(KB, \exists a \text{ Action}(a, 5))$

I.e., does KB entail any particular actions at $t = 5$?

Answer: *Yes*, $\{a/Shoot\} \leftarrow$ substitution (binding list)

Given a sentence S and a substitution σ ,

$S\sigma$ denotes the result of plugging σ into S ; e.g.,

$S = Smarter(x, y)$

$\sigma = \{x/Hillary, y/Bill\}$

$S\sigma = Smarter(Hillary, Bill)$

$Ask(KB, S)$ returns some/all σ such that $KB \models S\sigma$

Knowledge base for the wumpus world

“Perception”

$\forall b, g, t \text{ Percept}([Smell, b, g], t) \Rightarrow Smelt(t)$

$\forall s, b, t \text{ Percept}([s, b, Glitter], t) \Rightarrow AtGold(t)$

Reflex: $\forall t \text{ AtGold}(t) \Rightarrow \text{Action}(Grab, t)$

Reflex with internal state: do we have the gold already?

$\forall t \text{ AtGold}(t) \wedge \neg Holding(Gold, t) \Rightarrow \text{Action}(Grab, t)$

$Holding(Gold, t)$ cannot be observed

\Rightarrow keeping track of change is essential

Deducing hidden properties

Properties of locations:

$$\forall x, t \text{ } At(Agent, x, t) \wedge Smelt(t) \Rightarrow Smelly(x)$$

$$\forall x, t \text{ } At(Agent, x, t) \wedge Breeze(t) \Rightarrow Breezy(x)$$

Squares are breezy near a pit:

Diagnostic rule—infer cause from effect

$$\forall y \text{ } Breezy(y) \Rightarrow \exists x \text{ } Pit(x) \wedge Adjacent(x, y)$$

Causal rule—infer effect from cause

$$\forall x, y \text{ } Pit(x) \wedge Adjacent(x, y) \Rightarrow Breezy(y)$$

Neither of these is complete—e.g., the causal rule doesn't say squares far away from pits can be breezy

Definition for the *Breezy* predicate:

$$\forall y \text{ } Breezy(y) \Leftrightarrow [\exists x \text{ } Pit(x) \wedge Adjacent(x, y)]$$

Keeping track of change

Facts hold in situations, rather than eternally

E.g., $Holding(Gold, Now)$ rather than just $Holding(Gold)$

Situation calculus is one way to represent change in FOL:

Adds a situation argument to each non-eternal predicate

E.g., Now in $Holding(Gold, Now)$ denotes a situation

Situations are connected by the *Result* function

$Result(a, s)$ is the situation that results from doing a in s

Describing actions I

“Effect” axiom—describe changes due to action

$$\forall s \text{ } AtGold(s) \Rightarrow Holding(Gold, Result(Grab, s))$$

“Frame” axiom—describe **non-changes** due to action

$$\forall s \text{ } HaveArrow(s) \Rightarrow HaveArrow(Result(Grab, s))$$

Frame problem: find an elegant way to handle non-change

(a) representation—avoid frame axioms

(b) inference—avoid repeated “copy-overs” to keep track

Qualification problem: true descriptions of real actions require end
what if gold is slippery or nailed down or . . .

Ramification problem: real actions have many secondary cons
what about the dust on the gold, wear and tear on gloves, . . .

Describing actions II

Successor-state axioms solve the representational frame problem

Each axiom is “about” a **predicate** (not an action per se):

$$\begin{aligned} P \text{ true afterwards} \quad \Leftrightarrow \quad & [\text{an action made } P \text{ true} \\ & \vee \quad P \text{ true already and no action made } P \text{ false}] \end{aligned}$$

For holding the gold:

$$\begin{aligned} \forall a, s \quad & Holding(Gold, Result(a, s)) \Leftrightarrow \\ & [(a = Grab \wedge AtGold(s)) \\ & \vee (Holding(Gold, s) \wedge a \neq Release)] \end{aligned}$$

Making plans

Initial condition in KB:

$At(Agent, [1, 1], S_0)$

$At(Gold, [1, 2], S_0)$

Query: $Ask(KB, \exists s \text{ Holding}(Gold, s))$

i.e., in what situation will I be holding the gold?

Answer: $\{s / Result(Grab, Result(Forward, S_0))\}$

i.e., go forward and then grab the gold

This assumes that the agent is interested in plans starting at S_0 and that S_0 is the only situation described in the KB

Making plans: A better way

Represent plans as action sequences $[a_1, a_2, \dots, a_n]$

$PlanResult(p, s)$ is the result of executing p in s

Then the query $Ask(KB, \exists p \text{ Holding}(Gold, PlanResult(p, s)))$ has the solution $\{p/[Forward, Grab]\}$

Definition of $PlanResult$ in terms of $Result$:

$$\forall s \text{ } PlanResult([], s) = s$$

$$\forall a, p, s \text{ } PlanResult([a|p], s) = PlanResult(p, Result(a, s))$$

Planning systems are special-purpose reasoners designed to do this inference more efficiently than a general-purpose reasoner

Summary

First-order logic:

- objects and relations are semantic primitives
- syntax: constants, functions, predicates, equality, quantifiers

Increased expressive power: sufficient to define wumpus world

Situation calculus:

- conventions for describing actions and change in FOL
- can formulate planning as inference on a situation calculus