Assignment 1 Solutions EECS 4401/5326 Winter 2022

1-

a) $\forall x(Member(x) \equiv x = Joe \lor x = Sally \lor x = Bill \lor x = Ellen)$ ①

$MarriedTo(Joe, Sally)$ ②

$Brother(Ellen, Bill)$ ③

$\forall x \forall y(Member(x) \land MarriedTo(x,y) \supset Member(y))$ ④

b) We show that this set S does not entail $\neg \exists x\, MarriedTo(Ellen, x)$

Take $\mathcal{A} = \langle D, I \rangle$

$D = \{Joe, Sally, Bill, Ellen\}$

$I(Joe) = Joe$ & similarly for other constants

$I(Member) = D$

$I(MarriedTo) = \{\langle Joe, Sally\rangle, \langle Sally, Joe\rangle, \langle Bill, Ellen\rangle, \langle Ellen, Bill\rangle\}$

$I(Brother) = \{\langle Ellen, Bill\rangle\}$

Then $\mathcal{A} \models S$ but $\mathcal{A} \not\models \neg\exists x\, MarriedTo(Ellen, x)$

c) Add

$\forall x \forall y(Brother(x,y) \supset \neg MarriedTo(x,y))$ ⑤

$\forall x \forall y(MarriedTo(x,y) \supset MarriedTo(y,x))$ ⑥

$\forall x(\neg MarriedTo(x,x))$ ⑦

$\forall x \forall y \forall z(MarriedTo(x,y) \land MarriedTo(x,z) \supset y = z)$ ⑧

$x \neq y$ for distinct $x, y \in \{Joe, Sally, Bill, Ellen\}$ ⑨ UNA

Call this S'

Take arb $\mathcal{A} = \langle D, I \rangle$ assume $\mathcal{A} \models S'$.

By (1) and UNA $I(Member) = \{I(Joe), I(Sally), I(Bill), I(Ellen)\}$

By (2) (6) and (8) and UNA $\mathcal{A} \models \neg MarriedTo(Ellen, Sally)$

By (2) and (6) and (8) and UNA $\mathcal{A} \models \neg marriedTo(Ellen, Joe)$

By (5) and (3) $\mathcal{A} \models \neg marriedTo(Ellen, Bill)$

By (7) $\mathcal{A} \models \neg marriedTo(Ellen, Ellen)$

Thus $\mathcal{A} \models \forall x (Member(x) \supset \neg MarriedTo(Ellen, x))$ (*)

Suppose there is $a \in D$ s.t. $\mathcal{A}, \mu \models MarriedTo(Ellen, x)$ $\mu(x) = a$ (Ass)

we have $\mathcal{A} \models Member(Ellen)$ by (1)

so by (4) $\mathcal{A}, \mu \models Member(x)$

This together with * contradicts (Ass)

So $S' \models \neg \exists x\, MarriedTo(Ellen, x)$

2-

Show

$$\models \exists x \forall y \forall z ((P(y) \supset Q(z)) \supset (P(x) \supset Q(x)))$$

Negate query

$$\forall x \exists y \exists z \neg ((P(y) \supset Q(z)) \supset (P(x) \supset Q(x)))$$

Convert to clausal form

$$\forall x \exists y \exists z \neg (\neg(\neg P(y) \vee Q(z)) \vee (\neg P(x) \vee Q(x)))$$

$$\equiv \forall x \exists y \exists z ((\neg P(y) \vee Q(z)) \wedge (P(x) \wedge \neg Q(x)))$$

skolemize to

$$\forall x (\neg P(f(x)) \vee Q(g(x)))) \wedge$$
$$P(x) \wedge \neg Q(x)$$

Resolution Proof

1 $P(y)$

2 $\neg Q(z)$

3 $\neg P(f(x)) \vee Q(g(x))$

Conversion to clausal form yields 3 clauses

4 $Q(g(x)$ $R[1, 3a]\ \theta = \{y = f(x)\}$

5 $\square$ $R[2, 4]\ \theta = \{z = g(x)\}$

3a) It is satisfiable.

Let I be the interpretation $\langle \Delta_I, \cdot^I \rangle$ with

$\Delta_I = \{d\}$, $A^I = B^I = C^I = D^I = E^I = r^I = \emptyset$

Clearly $I \models C \sqsubseteq B$ since $C^I \subseteq B^I$ as $C^I = \emptyset$.

Similarly, I satisfies the other 4 axioms in T.

b) D is satisfiable wrt T.

Let I be the interpretation $\langle \Delta_I, \cdot^I \rangle$ with

$\Delta_I = \{d\}$, $A^I = B^I = C^I = E^I = r^I = \emptyset$ and $D^I = \{d\}$.

Clearly the first 4 axioms are satisfied because the ~~ex~~tension of the lefthand side is empty so the $\sqsubseteq$ is satisfied.

$I \models D \sqsubseteq \neg C$ since $D^I \subseteq (\neg C)^I$

as $\{d\} \subseteq \Delta_I \setminus \emptyset$.

also D^I is nonempty

c) $C \sqcap D$ is not satisfiable wrt T.

Since T contains $D \sqsubseteq \neg C$ we must have

$D^I \subseteq \Delta_I \setminus C^I$ and thus $D^I \cap C^I = \emptyset$

Any interpretation I that satisfies T cannot make $(C \sqcap D)^I$ non empty.

4 a)

$x_0 \cdot \{C_0\}$ where $C_0 = (\forall R.\forall R.\forall R \forall R.\neg A) \sqcap \quad = C_1$

$(\forall R.\exists R.\forall R.\exists R.B) \sqcap \quad = C_2$

$(\exists R.\forall R.\exists R.\exists R.C) \sqcap \quad = C_3$

$(\forall R.\forall R.\forall R.\exists R.A) \quad = C_4$

$\xrightarrow[\sqcap]{}$ 4 times

$x_0 \cdot \{C_0, C_1, C_2, C_3, C_4\}$

$\xrightarrow[\exists]{}$ $x_0 \cdot$ as above

$\quad | R$

$x_1 \cdot \{\forall R.\exists R.\exists R.C\}$

$\xrightarrow[\forall]{}$ 3 times

$x_0 \cdot$ as above

$\quad | R$

$x_1 \cdot \{\underbrace{\forall R.\exists R.\exists R.C}_{=C_5}, \underbrace{\forall R \forall R \forall R \neg A}_{=C_6},$
$\underbrace{\exists R \forall R \exists R.B}_{=C_7}, \underbrace{\forall R \forall R \exists R A}_{=C_8}\}$

$\xrightarrow[\exists]{}$ $x_0 \cdot$ as above

$\quad | R$

$x_1 \cdot \{C_5, C_6, C_7, C_8\}$

$\quad |$

$x_2 \cdot \{\forall R \exists R B\}$

$\xrightarrow[\forall]{}$ 3 times

$x_0 \cdot$ as above

$\quad | R$

$x_1 \cdot$ as above

$\quad |$

$x_2 \cdot \{\forall R \exists R B, \exists R \exists R C, \forall R \forall R \neg A,$
$\forall R \exists R A\}$

$\xrightarrow{\exists}$

x_0 · as above

| R

x_1 · as above

| R

x_2 · as above

| R

x_3 · $\{\exists R C\}$

$\xrightarrow{\forall}$ 3 times

x_0 · as above

| R

x_1 · as above

| R

x_2 · as above

| R

x_3 · $\{\exists R C, \exists R B, \forall R \neg A, \exists R A\}$

$\xrightarrow{\exists}$

x_0 · as above

| R

x_1 · as above

| R

x_2 · as above

| R

x_3 · as above

| R

x_4 · $\{A\}$

$\xrightarrow{\forall}$

x_0 · as before

| R

x_1 · as before

| R

x_2 · as before

| R

x_3 · as before

| R

x_4 · $\{A, \neg A\}$

X clash

Unsatisfiable

4 b) call the concept C_0

$x_0 \bullet \{C_0\}$

$\xrightarrow[\sqcap]{}$ 3 times

$$x_0 \bullet \{C_0, \overbrace{\forall R \exists R(\forall R.A \sqcup \forall R.B \sqcup \forall R.C)}^{C_1}, \overbrace{\forall R \forall R.(\exists R \neg A \sqcup \exists R \neg B)}^{C_2},$$
$$\overbrace{\forall R \forall R \exists R \neg C}^{C_3}, \exists R.C\}$$

$\xrightarrow[\exists]{}$

$x_0 \bullet \{C_0, C_1, C_2, C_3, \exists R.C\}$

$\quad | \; R$

$x_1 \bullet \{C\}$

$\xrightarrow[\forall]{}$

$x_0 \bullet$ as above

$\quad | \; R$

$$x_1 \bullet \{C, \overbrace{\exists R(\forall RA \sqcup \forall RB \sqcup \forall RC)}^{C_4}, \overbrace{\forall R(\exists R \neg A \sqcup \exists R \neg B)}^{C_5}, \overbrace{\forall R \exists R \neg C}^{C_6}\}$$

$\xrightarrow[\exists]{}$

$x_0 \bullet$ as above

$\quad | \; R$

$x_1 \bullet \{C, C_4, C_5, C_6\}$

$\quad | \; R$

$x_2 \bullet \{\forall RA \sqcup \forall R.B \sqcup \forall R.C\}$

$\xrightarrow[\forall]{}$ twice

$x_0 \bullet$ as above

$\quad | \; R$

$x_1 \bullet$ as above

$\quad | \; R$

$$x_2 \bullet \{\overbrace{\forall RA \sqcup \forall RB \sqcup \forall R.C}^{C_7}, \overbrace{\exists R \neg A \sqcup \exists R \neg B}^{C_8}, \overbrace{\exists R \neg C}^{C_9}\}$$

$\overrightarrow{\exists}$

x_0 as above

R

x_1 as above

R

x_2 as above

R

x_3 $\{\neg C\}$

$\overrightarrow{\sqcup \exists}$

x_0 as above

R

x_1 as above

R

x_2 $\{C_7, C_8, C_9, \exists R \neg A\}$

R R

x_3 $\{\neg C\}$ x_4 $\{\neg A\}$

or

x_0 as above

R

x_1 as above

x_2 $\{C_7, C_8, C_9, \exists R \neg B\}$

x_3 $\{\neg C\}$ x_4 $\{\neg B\}$

$\overrightarrow{\sqcup \forall}$

x_0 as above

x_1 as above

x_2 $\{C_7, C_8, C_9, \exists R \neg A, \forall R B\}$

x_3 $\{\neg C, B\}$ x_4 $\{\neg A, B\}$

✓

or

x_0 as above

x_1 as above

x_2 $\{C_7, C_8, C_9, \exists R \neg B, \forall R A\}$

x_3 $\{\neg C, A\}$ x_4 $\{\neg B,$

✓

$D = \{x_0, x_1, x_2, x_3, x_4\}$

$I(R) = \{\langle x_0, x_1\rangle, \langle x_1, x_2\rangle, \langle x_2, x_3\rangle, \langle x_2, x_4\rangle\}$

$I(A) = \{\}$

$I(B) = \{x_3, x_4\}$

$I(C) = \{\}$

Same except

$I(A) = \{x_3, x_4\}$

$I(B) = \{\}$