Overview (1):
- Before We Begin
  - Administrative details
  - Review → some questions to consider
- The Fourier Transform
  - Introduction
  - Background
- The One-Dimensional Fourier Transform
  - Introduction
  - Properties

Overview (2):
- The Two-Dimensional Fourier Transform
  - Introduction
  - Relationships
  - Properties
  - Filtering in the Frequency Domain
    - Properties of the frequency domain

Before We Begin

Administrative Details (1):
- Lab Six Today
  - We will continue with Lab 6 today
  - Lab report required
  - Requires the use of Matlab
  - No camera required
  - Ideally, you will read and look over the lab before coming to the lab!

Some Questions to Consider (1):
- What is the gradient operator?
- What is a first-order derivative?
- What is a second-order derivative?
- What is a Sobel operator?
- How do we apply the Sobel operator?
The Fourier Transform

Background (1):
- Fourier Domain Processing is Fundamental to Image Processing
  - To fully understand image processing at the very least, a basic understanding of Fourier processing is needed!
    - Perform a Fourier transform on (spatial domain) image to obtain its spectral components
    - Perform some operation on this spectral representation
    - Perform inverse Fourier operation to get back the spatial representation

Background (2):
- Introduced by the French mathematician Jean Baptiste Fourier in 1807
  - Published his theory in a book titled "The Theory of Heat" (1822)
  - Fourier's theory (Fourier series) → any function that periodically repeats itself (infinitely) can be expressed as a sum of sines and/or cosines of different frequencies, each multiplied by different coefficient
    - Doesn't matter how complicated the function is, as long as it repeats itself!

Background (3):
- Graphical Illustration

Background (4):
- Can Even Represent Non-Periodic, Finite Functions as the Integral of Sines and/or Cosine Functions
  - Provided area under resulting curve of the function is finite
  - This formulation is known as a Fourier transform as opposed to a Fourier series
  - Even more useful when considering practical problems → many times functions (signals) in "real-life" are not periodic and are finite

Background (5):
- Important Characteristics of Both Fourier Transform and Fourier Series
  - Can completely recover (reconstruct) the original (spatial representation) function with NO loss of information
  - Can work in the Fourier Domain and then return back to spatial domain → many problems are easier solved in the Fourier domain
Background (6):

- **The Functions (Images) we are Dealing with Are Finite in Duration**
  - We are therefore primarily interested and will be dealing with, is the Fourier transform
- **Many Image Enhancement Techniques in the Fourier Domain**
  - Extremely useful
  - Can be easier to understand what exactly is happening and how the operations work

Introduction (1):

- **Originally, Fourier Transform was Formulated with Continuous Time Signals**
  - We are dealing with sampled images
  - Finite intensity values and finite in duration
  - In other words, we are dealing with a discrete signal → remember, an image itself is a signal as in your DSP course, except we are now dealing with a two-dimensional signal as opposed to a one-dimensional signal you are familiar with
  - **Discrete Fourier Transform (DFT)** introduced to handle discrete signals

Discrete Fourier Transform (1):

- **One of the Most Common and Powerful Procedures Encountered in the Field of Digital Signal Processing in General**
  - Enables us to analyze, manipulate and synthesize signals in ways not possible with continuous (analog) signal processing
  - Used in every field of engineering
  - A solid understanding of the DFT is extremely important!

Discrete Fourier Transform (2):

- **What is the Discrete Fourier Transform ?**
  - A mathematical procedure used to determine the frequency (or harmonic) content of a discrete signal
  - Remember → discrete signal obtained by periodically sampling a continuous time signal in the time domain
  - Based on the Continuous Fourier Transform (CFT), denoted by X(f) (or F(u))

\[
X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt
\]

Discrete Fourier Transform (3):

- **Let's Analyze This Expression:**
  - f → frequency (spectral component)
  - x(t) → continuous time domain signal
  - \(e^{j2\pi ft}\) → a sinusoid (sine wave) of frequency f
  - In words → Fourier Transform of frequency component f is a correlation of the infinite input signal at each time step with a sine wave of frequency f \(\rightarrow X(f)\) tells us “how much” of the sine wave of frequency f the signal contains

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Discrete Fourier Transform (4): Discrete Fourier Transform (DFT)
Mathematically
\[ X[m] = \frac{1}{M} \sum_{n=0}^{M-1} x[n] e^{-j2\pi nm/M} \]
- Using Euler's Relationship \( e^{j\theta} = \cos(\theta) - j\sin(\theta) \) we obtain:
\[ X[m] = \frac{1}{M} \sum_{n=0}^{M-1} x[n](\cos(2\pi nm/M) - j\sin(2\pi nm/M)) \]

Discrete Fourier Transform (5):
\[ X[m] = \sum_{n=0}^{M-1} x[n](\cos(2\pi nm/M) - j\sin(2\pi nm/M)) \]
- \( X[m] \rightarrow \text{mth DFT output e.g., } X[0], X[1], \ldots X[M-1] \)
- \( m \rightarrow \text{index of the DFT output in frequency domain} \)
(\( m = 0, 1, 2, \ldots M-1 \))
- \( x[n] \rightarrow \text{sequence of input (discrete) samples} \)
(\( x[0], x[1], x[2], \ldots x[M-1] \))
- \( n \rightarrow \text{(discrete) time domain index of input samples} \)
- \( j = \sqrt{-1} \) (remember, complex numbers!)
- \( M \rightarrow \text{number of samples (same for input and DFT)} \)

Discrete Fourier Transform (6):
- Some Notes Regarding the DFT
- Indices for input samples and DFT output samples always go from 0 to M-1
- With M input time domain samples, the DFT determines the spectral content of the input at M equally spaced frequency points
- \( M \) is an important parameter and determines
  1. How many input samples are needed
  2. Resolution of the frequency domain results
  3. Amount of processing time required to calculate an M-point DFT

Discrete Fourier Transform (7):
- Some Notes Regarding the DFT (cont...)
- In words:
  - Each \( X[m] \) DFT output is the sum of a point for point product between an input sequence of input values and a complex sinusoid of the form \( \cos(\theta) - j\sin(\theta) \)
  - Exact frequencies of the different sinusoids depend on sample rate \( f_s \) and number of samples \( M \)
  - Fundamental frequency of the sinusoids is \( f_s /M \) and all other \( X[m] \) analysis frequencies are integer multiples of the fundamental!

Discrete Fourier Transform (8):
- Some Notes Regarding the DFT (cont...)
- The \( M \) separate DFT analysis frequencies are
\[ f_{\text{analysis}}(m) = \frac{mf_s}{M} \]
- So, \( X[0] \) gives us magnitude of an 0Hz ("DC") component contained in the signal, \( X[1] \) gives us magnitude of the fundamental component, \( X[2] \) gives us magnitude of 2 x fundamental component contained in signal etc.
- Finally, keep in mind, we are dealing with complex sinusoids \( \rightarrow \) magnitude and phase!

Discrete Fourier Transform (9):
- Determining the Magnitude and Phase Contained in each \( X[m] \) Term
- We can represent an arbitrary DFT output value \( X[m] \) by its real and imaginary parts
\[ X[m] = X_{\text{real}}[m] + jX_{\text{imag}}[m] = X_{\text{mag}}[m] \text{ at angle of } X_{\text{mag}}[m] \]
- The magnitude of \( X[m] \) is
\[ X_{\text{mag}}[m] = |X[m]| = \sqrt{X_{\text{real}}[m]^2 + X_{\text{imag}}[m]^2} \]
Discrete Fourier Transform (10):

- Determining the Magnitude and Phase
  Contained in each $X[m]$ Term (cont.)
  - The phase angle of $X[m]$, $\theta[m]$ is
    $$
    \theta[m] = \tan^{-1}\left(\frac{X_{\text{imag}}[m]}{X_{\text{real}}[m]}\right)
    $$
  - The power of $X[m]$, known as the power spectrum or spectral power is the magnitude squared
    $$
    X_{\text{ps}}[m] = X_{\text{mag}}[m]^2 = X_{\text{real}}[m]^2 + X_{\text{imag}}[m]^2
    $$

Discrete Fourier Transform (11):

- Graphical Illustration of Phase and Magnitude (Complex Plane)

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Some Properties of the 1D DFT

DFT Symmetry (1):

- Symmetry in DFT Output is Obvious!
  - Standard DFT is designed to accept complex input but most physical DFT inputs are "real" inputs
    - Non-zero real sample values
    - Imaginary values are assumed to be zero
  - With "real" input $x[n]$ the complex DFT outputs for $n = 1$ to $n = (M/2) - 1$ are redundant with frequency output values for $m > (M/2)$
    - $m$th DFT output will have the same value as the $(M-m)$th DFT output
    - the phase angle of the $m$th output is the negative of the $(M-m)$th DFT output

DFT Symmetry (2):

- Symmetry in DFT Output is Obvious! (cont.)
  - What does this symmetry mean?
    - If we perform an $M$-point DFT on a real input sequence, we get $M$ separate complex DFT output terms but only the first $M/2$ terms are independent
    - To obtain DFT of $x[n]$, we need only compute the first $M/2$ values of $X[m]$ where $0 \leq m < (M/2)$
    - The $X[M/2]$ to $X[M-1]$ DFT output terms provide no additional information about the spectrum of the real sequence $x[n]$

DFT Linearity (3):

- DFT is Linear
  - The DFT of the sum of two signals is equal to the sum of the transforms of each signal
    - Let $x_1[n]$ and $x_2[n]$ be two discrete input signals with DFT $X_1[m]$ and $X_2[n]$ respectively
    - Consider the sum of these two signals
      $$
      x_{\text{sum}}[n] = x_1[n] + x_2[n]
      $$
    - The DFT of $x_{\text{sum}}[n]$ is
      $$
      X_{\text{sum}}[m] = X_1[m] + X_2[m]
      $$

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DFT Linearity (4):
- DFT is Linear (cont...)
  - Exercise:
    - Mathematically prove this linearity property for the DFT

Inverse DFT

Inverse DFT - IDFT (1):
- Reverse the DFT Process
  - DFT transforms time-domain data into frequency domain representation
  - With inverse DFT, we transform frequency domain representation into time-domain representation
  - Perform IDFT on $X[m]$ frequency domain values

$$x[n] = \sum_{m=0}^{N-1} X[m](\cos(2\pi nm/N) - j\sin(2\pi nm/N))$$

Introduction to the Two-Dimensional Fourier Transform

Introduction (1):
- Straightforward to Extend One-Dimensional DFT to Two Dimensions
  - Two-dimensional DFT of a function (image) $f(x,y)$ of size $M \times N$ is given by

$$F[u,v] = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)e^{-j2\pi (ux/M + vy/N)}$$
  - Using Euler's relationship, we have the following

$$F[u,v] = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\cos(-j2\pi (ux/M + vy/N)) + j\sin(-j2\pi (ux/M + vy/N))$$

Introduction (2):
- Straightforward to Extend One-Dimensional DFT to Two Dimensions (cont...)
  - We can also easily extend the IDFT to two-dimensions as well. Given $F[u,v]$, IDFT is

$$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F[u,v]e^{j2\pi (ux/M + vy/N)}$$
  - Using Euler's relationship, we have the following

$$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F[u,v]\cos(-j2\pi (ux/M + vy/N)) + j\sin(-j2\pi (ux/M + vy/N))$$

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Introduction (3):
- Some Notes About 2D DFT
  - $x = 0, 1, 2, ..., M-1$ and $y = 0, 1, 2, ..., N-1$
  - Variables $u$ and $v$ are the transform or frequency variables and $x$, $y$ are the spatial or image variables
  - As with 1D DFT, we can define the magnitude, phase and power spectrum in a similar manner
  - Magnitude
    \[
    |F[u,v]| = \sqrt{R[u,v]^2 + I[u,v]^2}
    \]

Introduction (4):
- Some Notes About 2D DFT (cont...)
  - Phase $\phi[u,v]$
    \[
    \phi[u,v] = \tan^{-1}\frac{I[u,v]}{R[u,v]}
    \]
  - Power spectrum $P[u,v]$
    \[
    \]
  - where $R[u,v]$ and $I[u,v]$ are the real and imaginary components of the DFT $F[u,v]$ respectively

Introduction (5):
- Some Notes About 2D DFT (cont...)
  - Typically we multiply input image by $(-1)^{x+y}$ (pixel-by-pixel multiplication) prior to computing the DFT
  - Shifts the origin of the DFT to frequency coordinates $(M/2, N/2)$ → the center of the $M \times N$ 2D DFT
  - $M$, $N$ → even integers
  - After the multiplication, the DFT becomes
    \[
    F[u,v] = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f[x,y] e^{-j2\pi (uM+xN+y)/MN} (-1)xy
    \]

Introduction (6):
- Some Notes About 2D DFT (cont...)
  - Which is equal to
    \[
    F[u,v] = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} (f[x,y] e^{-j2\pi (uM+xN+y)/MN} (-1)xy
    \]
  - When we implement 2D DFT summations run from $u = 1$ to $M$ and $v = 1$ to $N$.
  - The center of the transform is at $u = (M/2) + 1$ and $v = (N/2) + 1$

Introduction (7):
- DC Component
  - DFT at the origin $(0,0)$ in the frequency domain is equal to the average gray level (intensity) of image $f(x,y)$
    \[
    F[0,0] = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)
    \]

Some 2D DFT Relationships (1):
- Conjugate Symmetry
  - If image $f(x,y)$ is real, its Fourier transform is conjugate symmetric
    \[
    F(u,v) = F^*(-u,-v)
    \]
  - where "**" indicates standard conjugate operation on a complex number
  - This implies the spectrum of the Fourier transform is symmetric
    \[
    |F(u,v)| = |F(-u,-v)|
    \]
Some 2D DFT Relationships (2):

- Conjugate Symmetry (cont...)
  - Conjugate symmetry and centering property
    simplify the specification of circularly symmetric
    filters in the frequency domain

- Relationship Between Samples in the
  Frequency and Spatial Domains

\[ \Delta u = \frac{1}{M \Delta x} \text{ and } \Delta u = \frac{1}{N \Delta y} \]

- In other words \( \rightarrow \) inverse relationship between
  spatial and frequency domain resolution

2D DFT Example (1):

2D DFT of a “Simple” Image

- 20 x 40 rectangle superimposed on black
  background of size 512 x 512
- Image multiplied by \((-1)^{x+y}\) prior to computing DFT
  to center the spectrum in the frequency domain

2D DFT Example (2):

- Some Comments regarding the Example
  - Inverse spatial vs. frequency relationship
    - Separation of “spectrum zeros” in \( u \) direction is
      twice separation in \( v \) direction \( \rightarrow \) 1 to 2 size
      ratio of rectangle in the image
    - Spectrum was processed using log transform prior
      to displaying to enhance gray level
  - Recall, dynamic range of DFT is huge and if we
    didn't process it, little detail would be evident
  - Most DFT spectra are processed with the log
    transform prior to displaying