COSC 4111/5111 3.0—Winter 2006

Date: Posted Feb. 26, 2006
Due: TBA (Approximate shelf life=3 weeks)

Problem Set No. 2

Most problems are from "Computability", **Chapters 3, 7, 8**. All relate to material in said chapters.

(1) **Ch.3.** Nos. 6, 22, 23, 26, 29, and 30.

(2) (**Grad**) Express the projections K and L of $J(x, y) = (x + y)^2 + x$ in *closed form*—that is, without using $(\mu y)_{<z}$ or bounded quantification.

(*Hint.* Solve for x and y the *Diophantine equation* $z = (x + y)^2 + x$. The term $\lfloor \sqrt{z} \rfloor$ is involved in the solution.)

(3) **Ch.7.** Nos. 5, 6, 7, 8 (**Do <u>not</u> use the "Rice theorems" for r.e. or recursive sets in these exercises!**).

(4) (**Grad**) Prove that a recursively enumerable set of sentences $\mathcal{T}$ over a finitely generated language (e.g., like that of arithmetic) admits a recursive set of axioms, i.e., for some recursive Γ, $\mathcal{T} = \mathbf{Thm}_{\Gamma}$.

(*Hint.* Note that for any $\mathcal{A} \in \mathcal{T}$, any two sentences in the sequence

$$\mathcal{A}, \mathcal{A} \wedge \mathcal{A}, \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}, \ldots$$

are logically equivalent. Now see if Theorem 1 and its corollaries (section 8.2 in "Computability") can be of any help.)

(5) Prove that $\lambda x.A_x(2) \notin \mathcal{PR}$, where $\lambda nx.A_n(x)$ is **our version** of the Ackermann function. Your proof must follow this path:

   (a) Prove that $A_n(x) < A_x(2)$ a.e. with respect to x.

   (b) Using the previous result, prove that if $\lambda x.f(x) \in \mathcal{PR}$, then $f(x) < A_x(2)$ a.e.

   (c) Conclude the argument.

(6) Prove that if $\lambda \vec{y}.f(\vec{y}) \in \mathcal{P}$ and $Q(\vec{x}, z) \in \mathcal{P}_*$, then $Q(\vec{x}, f(\vec{y})) \in \mathcal{P}_*$. Keep in mind the definition ("1-point-rule")

$$Q(\vec{x}, f(\vec{y})) \stackrel{\text{Def}}{\equiv} (\exists z)(z = f(\vec{y}) \wedge Q(\vec{x}, z))$$

(7) Prove with the techniques of the Appendix to Ch. 3 (the web notes) that the graph of f is r.e. iff $f \in \mathcal{P}$.

(8) Using the above and closure properties of $\mathcal{P}_*$ (cf. posted Appendix) prove that $\mathcal{P}$ is closed under definition by so-called "positive cases" (these are r.e. cases). That is, if all the f_i are in $\mathcal{P}$, all the Q_i are in $\mathcal{P}_*$ and g below is a function, then $g \in \mathcal{P}$.

$$g(\vec{x}) = \begin{cases} f_1(\vec{x}) & \text{if } Q_1(\vec{x}) \\ f_2(\vec{x}) & \text{if } Q_2(\vec{x}) \\ \vdots & \vdots \\ f_k(\vec{x}) & \text{if } Q_k(\vec{x}) \\ \uparrow & \text{otherwise} \end{cases}$$

Hint. Use the previous exercise and work with the graph of g.

(9) (**Grad**) In problem 5 presumably you concluded that $\{x : \phi_x \in \mathcal{PR}\}$ is not r.e., that is, you cannot computably enumerate all the ϕ-indices that happen to describe just the primitive recursive functions.

Show here that you can do the 2nd best: You can enumerate a a proper subset of the set of all the ϕ-indices, and this subset happens to define all the primitive recursive functions.

That is, prove that there is a $F \in \mathcal{R}$ such that

(a) If $\lambda\vec{x}.g(\vec{x}) \in \mathcal{PR}$, then for some e, $F(e, \langle\vec{x}\rangle) = g(\vec{x})$ for all $\vec{x}$.

(b) For all e, $\lambda z.F(e, z) \in \mathcal{PR}$.

(*Hint.* Use off the shelf our arithmetisation tools from the Appendix, but see what happens if you drop all the codes e with $(e)_0 = 3$ (and correspondingly drop the predicate "$U(u)$" from the definition of $Tree(u)$ and $T^{(n)}$.)

(10) (**For all**) In problem (9) I said about F, "prove that there is a $F \in \mathcal{R}$".

Take (9) as proved. Now prove that given F's properties (a) and (b), "$F \in \mathcal{R}$" is as much as we can say: That is, $\lambda ez.F(e, z) \notin \mathcal{PR}$.