COSC 4111/5111; Solutions —Winter 2014

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Problem Set No. 3 —Solutions

This is not a course on *formal* recursion theory. Your proofs should be informal (but $\neq$ sloppy), correct, and informative (and if possible short). Please do not trade length for correctness or readability.

(1) **Without using Rice's theorem or lemma**, explore/prove

(a) the set $A = \{x : \text{ran}(\phi_x) \text{ has exactly five distinct elements}\}$ is not recursive. (I.e., "$x \in A$ is unsolvable"). Is it r.e.? **Why?**

Answer. Let us skip to proving non-r.e.-ness from which non recursiveness also follows:

Define

$$\xi(x, y) = \begin{cases} rem(y, 5) & \text{if } \phi_x(x) \not\downarrow \text{ in } \leq y \text{ steps} \\ y & \text{otherwise} \end{cases}$$

We know that $\xi \in \mathcal{R}$, so let $h \in \mathcal{PR}$ such that $\xi(x, y) = \phi_{h(x)}(y)$ for all x, y. Thus, by our familiar analysis (see case of $\{x : \phi_x$ is a constant function $\}$ in text/class notes),

$$\phi_{h(x)} = \begin{cases} \lambda y.rem(y, 5) & \text{if } x \in \overline{K} \\ 0, 1, 2, 3, 4, 0, 1, 2, 3, 4, \ldots y_0, y_0 + 1, y_0 + 2, \ldots & \text{otherwise} \end{cases}$$

where y_0 depends on x and is the *first* y-value such that $\phi_x(x) \downarrow$ in y steps. Clearly only the condition $x \in \overline{K}$ leads to a range of $\phi_{h(x)}$ with exactly 5 elements; the other condition ($x \in K$) leads to infinite range. Thus $\overline{K} \leq A$ via this h.

(b) the set $D = \{x : \phi_x \text{ is the characteristic function of some set}\}$ is not recursive. Is it r.e.? **Why?**.

Answer. We use the ψ defined for the case $\{x : \phi_x$ is a constant function$\}$ in the text/class notes.

$$\psi(x, y) = \begin{cases} 0 & \text{if } \phi_x(x) \not\downarrow \text{ in } \leq y \text{ steps} \\ \uparrow & \text{otherwise} \end{cases}$$

We know that $\psi \in \mathcal{P}$ using def. by pos. cases, so let $\sigma \in \mathcal{PR}$ such that $\psi(x,y) = \phi_{\sigma(x)}(y)$ for all x, y. Thus

$$\phi_{\sigma(x)} = \begin{cases} \lambda y.0 & \text{if } x \in \overline{K} \\ \underbrace{\langle 0, 0, \ldots, 0 \rangle}_{y_0 \text{ zeros}} & \text{otherwise} \end{cases}$$

where y_0 depends on x and is the *first* y-value such that $\phi_x(x) \downarrow$ in y steps. Clearly only the condition $x \in \overline{K}$ leads to a characteristic function (the one for $\mathbb{N}$); the other condition ($x \in K$) leads to a finite function which is NOT characteristic (char. functions are total). Thus $\overline{K} \leq D$ via this σ.

So D is neither r.e. nor recursive.

(c) the set $E = \{x : \text{ran}(\phi_x) \text{ contains only odd numbers}\}$ is not recursive. Is it r.e.? **Why?**

Answer. It is not r.e. hence nor recursive:

Define

$$g(x,y) = \begin{cases} 1 & \text{if } \phi_x(x) \not\downarrow \text{ in } \leq y \text{ steps} \\ 2 & \text{otherwise} \end{cases}$$

As we know from class, $g \in \mathcal{P}$, in fact, in $\mathcal{R}$. Thus, for some $\tau \in \mathcal{PR}$,

$$\phi_{\tau(x)} = \begin{cases} \lambda y.1 & \text{if } x \in \overline{K} \\ \langle \underbrace{1, \ldots, 1}_{y_0 \text{ ones}} 2, 2, \ldots \rangle & \text{otherwise} \end{cases} \tag{2}$$

where y_0 is the smallest number of steps it takes to have $\phi_x(x) \downarrow$. Only in the top case $\text{ran}(\phi_{\tau(x)})$ *contains only odd numbers*. Thus, $\overline{K} \leq E$.

(2) Prove that there is a function $f \in \mathcal{P}$ such that $W_x \neq \emptyset$ implies $f(x) \downarrow$ and $f(x) \in W_x$.

Hint. To define $f(x)$ you want, given the verifier x (for W_x), to dovetail its computation as follows: consider systematically all pairs $\langle y, z \rangle$ *until* $T(x,y,z)$ holds. If so, set $f(x) = y$ (if not, go happily forever; this is the case $W_x = \emptyset$). *Make this mathematically precise*!

Answer. Thanks for the hint :-)

So, here it goes:

$$f(x) = \Big((\mu z) T(x, (z)_0, (z)_1) \Big)_0$$ □

(3) Do Exercise 5.2.0.32, p.359.

In view of the bounding lemma, prove that *switch* (the "full" if-then-else) and max are *not* in $\mathcal{E}^0$.

Answer. If $sw \in \mathcal{E}^0$ then one of the following must hold:

- For some k, $sw(x, y, z) \le x + k$ for all x, y, z. Take $y = k + 1$ and $x = 0$ to get a contradiction.
- For some k, $sw(x, y, z) \le y + k$ for all x, y, z. Take $z = y + k + 1$ and $x = 1$ to get a contradiction.
- For some k, $sw(x, y, z) \le z + k$ for all x, y, z. Take $y = z + k + 1$ and $x = 0$ to get a contradiction.

If $max \in \mathcal{E}^0$ then one of the following must hold:

- For some k, $\max(x, y) \le x + k$ for all x, y. Take $y = x + k + 1$ to get a contradiction.
- For some k, $\max(x, y) \le y + k$ for all x, y. Take $x = y + k + 1$ to get a contradiction.

(4) From Section 5.3 do Problem 23.

#23: Prove that $T \in \mathcal{E}^3_*$ and $d \in \mathcal{E}^3$.

Proof. We systematically scan the proof that $T \in \mathcal{PR}_*$ contained in the text and modify it to obtain this sharper result.

What properties and functions/predicates from $\mathcal{PR}/\mathcal{PR}_*$ did we use in the proof that

$$URM(z),\ Comp^{(n)}(z, y) \text{ —and therefore } T^{(n)}(x, \vec{y}_n, z) \tag{1}$$

are in $\mathcal{PR}_*$?

First of all, we used closure properties of $\mathcal{PR}_*$ (Boolean and bounded quantification) and of $\mathcal{PR}$, including closure under $(\mu y)_{\le z}$. Even though we used $(\mu y)_{\le z}$ in $\mathcal{PR}$, the favourite of the $\mathcal{E}^n$ classes —$(\mathring{\mu} y)_{\le z}$— works equally well as it can trivially be verified.

Key functions/predicates in the definition of the predicates in (1) were:

$\lambda xy.\lfloor x/y \rfloor$, exponentials ($x^y$) —in particular $\lambda nx.p_n^x$— prime-power coding / decoding and its tools: $[\cdots], Seq(z), lh, (z)_i$.

To get $\lambda nx.p_n^x$ in $\mathcal{E}^3$ is easy:

- Since + and times are in $\mathcal{E}^3$ (and earlier), $Pr(x)$ is in $\mathcal{E}^3_*$ by closure properties and the fact that $x|y$ is no more than $(\exists z)_{\le y} y = xz$.
- We get $\lambda n.p_n$ in $\mathcal{E}^3$, **the very same way** we did it for $\mathcal{PR}$: $\pi(x)$ first (the recursion is bounded —by x trivially; π is even in $\mathcal{E}^0$), then $y = p_n$ (in $\mathcal{E}^0_*$ and hence in $\mathcal{E}^3_*$), and then obtain $\lambda n.p_n$ as

$$p_n = (\mathring{\mu} y)_{\le 2^{2^{n+1}}} y = p_n$$

  Note that 2^x is in $\mathcal{E}^3$ as we have a well-known trivial recursion with iterator $x + y$, and 2^x is bounded by $A^k_2(x)$, for an appropriate k.
- Get x^y. Use the obvious recursion that is based on "times" (the latter already in $\mathcal{E}^2$), and note the bounding $x^y \le 2^{xy}$ —for a verification of the inequality note the equivalent inequality

$$y \log_2 x \le xy$$

  which is clearly true since $\log_2 x \le x$.
- Thus we have (omitting λ) p^z_n in $\mathcal{E}^3$ by substituting p_n into x in x^z.
- From the text, $\mathcal{E}^3$ is closed under $\sum_{\le z}$ and $\prod_{\le z}$ (5.2.0.33 ans 5.2.0.34, pp. 359–360)
- Armed with the above, $[x_0, x_1, \ldots, x_n] = \prod_{i \le n} p_i^{x_i+1}$ is in $\mathcal{E}^3$ for exactly the same reasons it is in $\mathcal{PR}$.
  Thus the following are also in $\mathcal{E}^3$ for exactly the same reasons they are in $\mathcal{PR}$ (proofs exactly as in the case of $\mathcal{PR}$ except that we now employ $(\mathring{\mu} y)_{\le z}$ rather than $(\mu y)_{\le z}$):
  (i) $\lambda xy.\lfloor x/y \rfloor$ —this is used in the $yield(z, u, v)$ predicate employed in the definition of $Comp^{(n)}(z, y)$
  (ii) $\lambda xy. \exp(x, y)$
  (iii) $\lambda xy.(y)_x$
  (iv) $\lambda x.lh(x)$

Now let us look at 2.3.03 first (primitive recursiveness of $URM(z)$). Given the above bullets the argument there shows **also** that $URM(z) \in \mathcal{E}^3_*$.

Turning to $Comp^{(n)}(z, y)$, we see no tool used there that we did not establish above as being in $\mathcal{E}^3$ or $\mathcal{E}^3_*$

In the proof of 2.3.0.7 (Kleene T-predicate) nothing new was done. So $T^{(n)} \in \mathcal{E}^3_*$. As for the decoding function d it is given by

$$d(y) = \big((y)_{lh(y) \dot{-} 1}\big)_1$$

and we are done by bullet (iv) above. □