York University
Department of Electrical Engineering and Computer Science
Lassonde School of Engineering

EECS1028Z FINAL TAKE-HOME EXAM, April 22, 2024; 2:00–4:00PM —SOLUTIONS

Professor George Tourlakis

Question 1. (a) (1 MARK) Define precisely the term "**Set** A **is Finite**".
Answer: $A = \emptyset$ **OR** $A \sim \{0, 1, \ldots, n\}$, that is, $A \sim \{x \in \mathbb{N} : x \leq n\}$. □

(b) (4 MARKS) Let $n \in \mathbb{N}$ and $n > 0$. Let $X \subseteq \{x \in \mathbb{N} : x \leq n\}$.
Prove that X is finite.

Proof. I argue **by contradiction**.

Assume that X is *infinite* instead.

But

$$X \subseteq \{x \in \mathbb{N} : x \leq n\} \subseteq \mathbb{N} \tag{1}$$

Then,

i. By a theorem from NOTES/Class, X being an infinite subset of $\mathbb{N}$ is *enumerable*, meaning:

$$X \sim \mathbb{N} \tag{2}$$

ii. Let $f : X \to \mathbb{N}$ be the 1-1 correspondence we have in mind in (2). Thus f is onto $\mathbb{N}$. Define $g : \{0, 1, 2, \ldots, n\} \to \mathbb{N}$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in X \\ \uparrow & \text{if } x \in \{0, 1, \ldots, n\} - X \end{cases} \tag{3}$$

g is onto $\mathbb{N}$ since its sub-function f (see definition in (3), that makes clear that $f \subseteq g$) already "covers" $\mathbb{N}$ with its outputs. So does g then!

But this contradicts another theorem from class (5.2.8) and we see that the "red" assumption above must be reversed! □

Question 2. (4 MARKS) Prove that an enumerable set is infinite.

Proof. Let A be enumerable. This means $A \sim \mathbb{N}$.

By contradiction, let A be also finite, hence $A \sim \{0, 1, \ldots, n\}$ for some n. Thus (using symmetry of $\sim$ (class; assignments)

$$\{0, 1, \ldots, n\} \sim A \sim \mathbb{N}$$

and by transitivity of $\sim$ (class; assignments/notes) $\{0, 1, \ldots, n\} \sim \mathbb{N}$ which is a **contradiction** since **no ONTO function** from the left set onto the right set is possible (Class NOTES; Corollary 5.2.8). □

Question 3. (3 MARKS) Prove that the set $\{1\}$ is countable.

Proof. Indeed, the function $f : \mathbb{N} \to \{1\}$ that for each $x \in \mathbb{N}$ returns "1" is onto the set $\{1\}$. By definition of countability $\{1\}$ is countable with *enumerating function* f. $\square$

Question 4. (a) (1 MARK) Prove that the class $\{7^m : m \geq 0\}$ is a **set**.

Proof. The set $\mathbb{N}$ is a labelling set for the class $\{7^m : m \geq 0\}$.
Each member 7^m is labelled by m. By Principle 3, $\{7^m : m \geq 0\}$ is a *set*. □

(b) (4 MARKS) Prove that the set $\{7^m : m \geq 0\}$ is **enumerable**.

Proof. Indeed we show that the function $f : \mathbb{N} \to \{7^m : m \geq 0\}$ given, for each $x \in \mathbb{N}$, by $f(x) = 7^x$ is 1-1, total and onto $\{7^m : m \geq 0\}$.

- **totalness**: For *each* $x \in \mathbb{N}$ —the **left field**— we *do have an output*: 7^x.
- **1-1ness**. What do we conclude from $f(x) = f(y)$?
  First we translate: It says $7^x = 7^y$. But 7 is a prime and by the "unique prime-factorisation theorem" of Euclid, the number (same on both sides of "=") has only one factorisation, so $x = y$. This proves 1-1ness.
- **ontoness**. Prove that any number 7^m in the right field of f —namely $\{7^x : x \in \mathbb{N}\}$— is the output of a "call" $f(x)$. Sure! $x = m$.

We proved that

$$\{7^x : x \in \mathbb{N}\} \overset{f}{\sim} \mathbb{N}$$

which by definition says that $\{7^x : x \in \mathbb{N}\}$ is enumerable. □

Question 5. (4 MARKS) Prove $\vdash (\exists x)(A \to B) \to (\forall x)A \to (\exists x)B$.

Proof. By DThm, prove instead

$$(\exists x)(A \to B), (\forall x)A \vdash (\exists x)B$$

Here it is:

1) $(\exists x)(A[x] \to B[x])$ $\langle$hyp$\rangle$
2) $(\forall x)A[x]$ $\langle$hyp via DThm$\rangle$
3) $A[c] \to B[c]$ $\langle$aux. hyp for line 1; c fresh; and not in conclusion$\rangle$
4) $A[c]$ $\langle 2 + \text{Spec}\rangle$
5) $B[c]$ $\langle 3 + 4 + \text{MP}\rangle$
6) $(\exists x)B[x]$ $\langle 5 + \text{Dual Spec}\rangle$

□

Question 6. (a) (2 MARKS) Let A be a formula of Predicate Logic. What does the notation "$A(x)$" mean **exactly**? ONE sentence please!

Answer. "$A(x)$" means that "x is the ONLY free variable in A". □

(b) (4 MARKS) Consider $(\exists x)A(x) \to A(x)$.
Show that it cannot possibly be valid, and **do so** by finding a simple formula A over $\mathbb{N}$ that provides a counterexample to validity.

Proof. By counterexample:

If the given is valid so is the special case over the natural numbers $\mathbb{N}$ below

$$(\exists x)x = 0 \to x = 0 \tag{1}$$

BUT: (1) is NOT true as required for all values of the free occurrence of (3rd) x. Indeed, consider the x-value 42:

$$\overbrace{(\exists x)x = 0}^{\mathbf{t}} \to \overbrace{42 = 0}^{\mathbf{f}} \tag{2}$$

□

Question 7. (4 MARKS) Use induction to prove that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{1}$$

Proof.

Basis. $n = 1$. We have $lhs = 1$ and $rhs = \dfrac{1 \times (1+1) \times (2 \times 1 + 1)}{6} = 1$. Equal.

I.H. Fix n and assume (1).

I.S. Prove the case where the n fixed above is replaced by $n + 1$.
Here it goes ("equationally" as in high school).

$$\begin{aligned}
\overbrace{1^2 + 2^2 + 3^2 + \cdots + n^2}^{I.H.\ applies} + (n+1)^2 &\overset{I.H.}{=} \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= (n+1)\frac{2n^2 + n + 6(n+1)}{6} \\
&= (n+1)\frac{2n^2 + 7n + 6}{6} \qquad (\ddagger)
\end{aligned}$$

Pause. Factoring the last numerator. By high school techniques first solve

$$2n^2 + 7n + 6 = 0$$

for n:

$$n = \begin{cases} \frac{-7+\sqrt{49-48}}{4} \\ \frac{-7-\sqrt{49-48}}{4} \end{cases} = \begin{cases} -6/4 \\ -2 \end{cases}$$

Thus

$$\frac{2n^2 + 7n + 6}{6} = 2(n+2)(n+6/4) = (n+2)(2n+3)$$

Subtituting the factorisation above for the last result ($\ddagger$) above we obtain

$$1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = (n+1)\frac{(n+2)(2n+3)}{6}$$

Noting that $n + 2 = (n+1) + 1$ and $2n + 3 = 2(n+1) + 1$ we have proved the I.S.! $\square$

Question 8. Consider the inductive definition of the set B as $\mathrm{Cl}(\mathcal{I}, \mathcal{O})$ —that is, we set $B = \mathrm{Cl}(\mathcal{I}, \mathcal{O})$— where

(a) $\mathcal{I} = \{\lambda\}$

(b) $\mathcal{O}$ contains two operations,

i. $(X, Y) \longrightarrow \boxed{concat} \longrightarrow XY$ **Comment**: Concatenation of X and Y in that order. and

ii. $X \longrightarrow \boxed{paren} \longrightarrow (X)$ **Comment**: Concatenation of "(", "X" and ")" in that order.

Prove:

- (3 MARKS) The strings

$$(), (()), \text{ and } ()(()) \text{ are in } B$$

Proof.

– *For* (). B contains λ (is in $\mathcal{I}$) and is closed under operation "*paren*". Thus the result of this operation on λ produces () in B.

– *For* (()). By the result in the above bullet, since $() \in B$, so is (()) as the result of *paren* is (()).

– *For* ()(()). By the results in the above two bullets, since $() \in B$, AND so is $(()) \in B$, then —since the result of *concat*, on inputs () and (()), is ()(())— we are done by closure of B under *concat*. □

- (4 MARKS) If $X \in B$, then X has as many left brackets as it has right brackets.

Proof. We do induction on the closure B to prove the "property": that any $X \in B$ "has as many left brackets as it has right brackets".

Basis. We verify the property for all the initial objects. There is only one such object (member of $\mathcal{I}$), namely, λ.

This indeed has 0 left and 0 right brackets. Equal number!

Propagation of the property —"lefts are exactly as many as rights"— by all operations. There are TWO operations only.

Op. **1** **Let** inputs X and Y of *concat* have the property. Now, the output is XY and clearly has as many lefts (X-lefts + Y-lefts) as it has rights (X-rights + Y-rights). Property propagates with *concat*.

Op. **2** **Let** input X of *paren* have the property of "lefts are in equal numbers as that of rights". But the output "(X)" of *paren* has the property too as we add ONE left and ONE right to those of X. Property propagates with *paren*.

Done. □