1. *Tutorial*

We do problems 3.8, 3.10, 3.13, 3.19, 3.23, 3.24, 3.26, 3.27 and 3.32 for extra practice, supplementing the examples worked out in the text (Chapter 3 of GS). Of course, we use our schemata-notation from class, so we prove in each case not just a single theorem (like $\neg\neg p \equiv p$), but a theorem-schema (like $\neg\neg A \equiv A$).

3.8 Prove $\neg\neg A \equiv A$ (with **no** additional assumptions).

We also often say "Prove that $\vdash \neg\neg A \equiv A$". Assuming that we know what we are doing, that is probably "OK". But strictly speaking, we should say "**meta**prove that $\vdash \neg\neg A \equiv A$", because "$\vdash \neg\neg A \equiv A$" says "$\neg\neg A \equiv A$ is a logical theorem". Note that the latter statement, although correct, is **not in its entirety** a formula of our language—only the "$\neg\neg A \equiv A$" part of it is—hence it cannot be a "theorem".

Now that we got the jargon straight, let us proceed:

$$
\begin{aligned}
&\neg\neg A \equiv A \\
= &\Big\langle \vdash \neg A \equiv B \equiv A \equiv \neg B \text{ was proved in class} \Big\rangle \\
&\neg A \equiv \neg A
\end{aligned}
$$

The last formula is a theorem (class: $\vdash A \equiv A$). Done.

Wait a minute! If we proved in class that "$\vdash A \equiv A$", how come this is as good as "$\vdash \neg A \equiv \neg A$"? □

3.10 Prove that $\vdash (A \not\equiv B) \equiv \neg A \equiv B$.

It is important to note that whenever a **defined** connective such as $\not\equiv, \Leftarrow, \not\Leftarrow, \not\Rightarrow$ is involved, we do ***not*** involve it in the formal proof but first ***translate*** the informally written formula to correct form, and ***only then*** start the proof.

So, ***translation***: We are ***really*** being asked to prove that:

$$\vdash \neg(A \equiv B) \equiv \neg A \equiv B$$

But this so! This is the axiom on distribution of $\neg$ over $\equiv$. □

3.13 Prove that $\vdash ((A \not\equiv B) \equiv C) \equiv (A \not\equiv (B \equiv C))$.

Translation: We are ***really*** being asked to prove that:

$$\vdash (\neg(A \equiv B) \equiv C) \equiv \neg(A \equiv (B \equiv C))$$

Here it goes:

$$
\begin{aligned}
& \neg(A \equiv B) \equiv C \\
= & \Big\langle \text{Axiom “distribution of } \neg \text{ over } \equiv \Big\rangle \\
& \neg\Big((A \equiv B) \equiv C\Big) \\
= & \Big\langle \text{Assoc. of } \equiv \text{ and Leib. with formula } \neg r \Big\rangle \\
& \neg\Big(A \equiv (B \equiv C)\Big)
\end{aligned}
$$

Done. $\Box$

3.19 Prove $\vdash A \vee B \equiv A \vee \neg B \equiv A$.

A common (fatal) error that I often see is the interpretation of the above as

$$
\begin{aligned}
& A \vee B \\
= & \\
& A \vee \neg B \\
= & \\
& A
\end{aligned}
$$

which is “way out”. None of these “=” holds!

Let’s do it then, pretending the rightmost $\equiv$ is the last one. (What do I mean by “the last one”? Can I do that?)

$$
\begin{aligned}
& A \vee B \equiv A \vee \neg B \\
= & \Big\langle \text{Axiom: distrib. of } \vee \text{ over } \equiv \Big\rangle \\
& A \vee (B \equiv \neg B) \\
= & \Big\langle \text{Leib. on } A \vee r \text{ plus } \vdash \neg A \equiv A \equiv \mathit{false} \text{ from class} \Big\rangle \\
& A \vee \mathit{false} \\
= & \Big\langle \text{Class: } \vdash A \vee \mathit{false} = A \Big\rangle \\
& A
\end{aligned}
$$

Done. $\Box$

3.23. Prove that $\vdash A \wedge A \equiv A$.

$$
\begin{aligned}
& A \wedge A \equiv A \\
= & \left\langle \text{By "GR"} \right\rangle \\
& A \equiv A \vee A
\end{aligned}
$$

The second line is an axiom (idempotent for $\vee$). Done. □

3.24. Prove that $\vdash A \wedge \mathit{false} \equiv \mathit{false}$.

$$
\begin{aligned}
& A \wedge \mathit{false} \equiv \mathit{false} \\
= & \left\langle \text{By "GR"} \right\rangle \\
& A \equiv A \vee \mathit{false}
\end{aligned}
$$

The second line is an (logical) theorem (done in class). Done. □

3.26. Prove that $A \wedge \neg A \equiv \mathit{false}$.

$$
\begin{aligned}
& A \wedge \neg A \\
= & \left\langle \text{By "GR"} \right\rangle \\
& A \equiv \neg A \equiv A \vee \neg A \\
= & \left\langle \text{By redundant true plus excluded middle axiom, using Leib. on } A \equiv \neg A \equiv r \right\rangle \\
& A \equiv \neg A \equiv \mathit{true} \\
= & \left\langle \text{By redundant true} \right\rangle \\
& A \equiv \neg A \\
= & \left\langle \text{By } \vdash A \equiv \neg A \equiv \mathit{false} \text{ from class} \right\rangle \\
& \mathit{false}
\end{aligned}
$$

Done. □

3.27. Prove that $\vdash A \wedge (A \vee B) \equiv A$.

$$\begin{aligned}
& A \wedge (A \vee B) \\
= & \Big\langle \text{By "GR"} \Big\rangle \\
& A \equiv A \vee B \equiv A \vee A \vee B \\
= & \Big\langle \text{Leib. on } A \equiv A \vee B \equiv r \vee B \text{ and idemp. axiom: } A \vee A \equiv A \Big\rangle \\
& A \equiv A \vee B \equiv A \vee B \\
= & \Big\langle \text{Redundant true and } \vdash A \equiv A \text{ (class) using Leib. on } A \equiv r \Big\rangle \\
& A \equiv \mathit{true} \\
= & \Big\langle \text{Redundant true} \Big\rangle \\
& A
\end{aligned}$$

Done. $\Box$

3.32. Prove that $\vdash \neg(A \wedge B) \equiv \neg A \vee \neg B$.

The only "trick" in the proof that follows is not a trick at all. We "factor" formulas (using distribution of $\vee$ over $\equiv$) just as we do so with numbers. In slow motion, ***compare***

$$\begin{aligned}
& a + a \times b \\
= & \Big\langle \vdash a \times 1 = a \text{ on numbers, i.e., "1" is the "}\times\text{-identity"} \Big\rangle \\
& a \times 1 + a \times b \\
= & \Big\langle \text{Distribution of } \times \text{ over } + \Big\rangle \\
& a \times (1 + b)
\end{aligned}$$

with

$$\begin{aligned}
& A \equiv A \vee B \\
= & \Big\langle \vdash A \vee \mathit{false} \equiv A \text{ on formulas, i.e., "}\mathit{false}\text{" is the "}\vee\text{-identity"} \Big\rangle \\
& A \vee \mathit{false} \equiv A \vee B \\
= & \Big\langle \text{Distribution of } \vee \text{ over } \equiv \Big\rangle \\
& A \vee (\mathit{false} \equiv B)
\end{aligned}$$

Ready for the main event (which uses the immediately above "factoring" twice):

$$
\begin{aligned}
&\neg(A \wedge B)\\
=&\Big\langle \text{By “GR” using Leib. on } \neg r \Big\rangle\\
&\neg(A \equiv B \equiv A \vee B)\\
=&\Big\langle \text{Distribution of } \neg \text{ over } \equiv \Big\rangle\\
&\neg A \equiv B \equiv A \vee B\\
=&\Big\langle \text{Leib. on } \neg A \equiv r \equiv A \vee B \text{ and } \vdash \mathit{false} \vee B \equiv B \text{ (class)} \Big\rangle\\
&\neg A \equiv \mathit{false} \vee B \equiv A \vee B\\
=&\Big\langle \text{Leib. on } \neg A \equiv r \text{ and distrib. of } \vee \text{ over } \equiv \Big\rangle\\
&\neg A \equiv (\mathit{false} \equiv A) \vee B\\
=&\Big\langle \text{Leib. on } \neg A \equiv r \vee B \text{ and } \vdash \mathit{false} \equiv A \equiv \neg A \text{ (class)} \Big\rangle\\
&\neg A \equiv \neg A \vee B\\
=&\Big\langle \text{Leib. on } r \equiv \neg A \vee B \text{ and } \vdash \mathit{false} \vee B \equiv B \text{ (class)} \Big\rangle\\
&\neg A \vee \mathit{false} \equiv \neg A \vee B\\
=&\Big\langle \text{Distrib. of } \vee \text{ over } \equiv \Big\rangle\\
&\neg A \vee (\mathit{false} \equiv B)\\
=&\Big\langle \text{Leib. on } \neg A \vee r \text{ and } \vdash \mathit{false} \equiv A \equiv \neg A \text{ (class)} \Big\rangle\\
&\neg A \vee \neg B
\end{aligned}
$$

Done. $\Box$