Facts-List

Here I list some things from class that can be used "off the shelf", without proof (even if they are theorems or metatheorems).

The axiom schemata (1)–(12) below are "good" ONLY for those questions that you are asked to do with "***Chapter 3 Techniques***".

In your answers please refer to them **by name** or by the **numbers given here** (*not by numbers in the text*!).

Associativity of $\equiv$	$((A \equiv B) \equiv C) \equiv (A \equiv (B \equiv C))$	(1)
Symmetry of $\equiv$	$(A \equiv B) \equiv (B \equiv A)$	(2)
true: **Identity of** $\equiv$	$\mathit{true} \equiv A \equiv A$	(3)
false: **Negation of** *true*	$\mathit{false} \equiv \neg \mathit{true}$	(4)
Distributivity of $\neg$ **over** $\equiv$	$\neg(A \equiv B) \equiv \neg A \equiv B$	(5)
Associativity of $\vee$	$(A \vee B) \vee C \equiv A \vee (B \vee C)$	(6)
Symmetry of $\vee$	$A \vee B \equiv B \vee A$	(7)
Idempotency of $\vee$	$A \vee A \equiv A$	(8)
Distributivity of $\vee$ **over** $\equiv$	$A \vee (B \equiv C) \equiv A \vee B \equiv A \vee C$	(9)
Excluded Middle	$A \vee \neg A$	(10)
Golden Rule	$A \wedge B \equiv A \equiv B \equiv A \vee B$	(11)
Implication	$A \Rightarrow B \equiv A \vee B \equiv B$	(12)

The following metatheorems are good for ***both*** Propositional (Ch. 3–4) and Predicate Calculus (Ch. 8–9):

1. *Redundant True.* $\Gamma \vdash A$ iff $\Gamma \vdash A \equiv \mathit{true}$
2. *Modus Ponens (MP).* $A, A \Rightarrow B \vdash B$
3. *Cut Rule.* $A \vee B, \neg A \vee C \vdash B \vee C$
4. *Deduction Theorem.* If $\Gamma, A \vdash B$, then $\Gamma \vdash A \Rightarrow B$
5. *Proof by contradiction.* $\Gamma, \neg A \vdash \mathit{false}$ iff $\Gamma \vdash A$
6. *Post's Theorem.* (Also called "tautology theorem", or even "completeness of Propositional Calculus theorem")

   If $\models_{\text{taut}} A$, then $\vdash A$.

   Also: If $B_1, \ldots, B_n \models_{\text{taut}} A$, then $B_1, \ldots, B_n \vdash A$
7. *Proof by cases.* $A \Rightarrow B$, $C \Rightarrow D \vdash A \vee C \Rightarrow B \vee D$

   Also the special case: $A \Rightarrow B$, $C \Rightarrow B \vdash A \vee C \Rightarrow B$

The following are the axioms for Ch.9 and 8: ***Any partial generalization of any formula in groups* Ax1–Ax6 *is an axiom for Predicate Calculus.***

Groups **Ax1–Ax6** contain:

Ax1. All tautologies.

Ax2. For every formula A, $(\forall x)A \Rightarrow A[x := t]$, for any term t.

Ax3. For every formula A and variable x *not free in* A, the formula $A \Rightarrow (\forall x)A$.

Ax4. For every formulas A and B, $(\forall x)(A \Rightarrow B) \Rightarrow (\forall x)A \Rightarrow (\forall x)B$.

Ax5. For *each* object variable x, the formula $x \approx x$.

Ax6. (*Leibniz's characterization of equality—1st order version. "3.83"*) For any formula A, any object variable x and any terms t, s, the formula $t \approx s \Rightarrow (A[x := t] \equiv A[x := s])$.

Primary rules of inference are **Equanimity**, **Transitivity** and **PSL** in ***both*** Ch.3 and Ch.9.

$$\frac{A \equiv B}{C[p := A] \equiv C[p := B]}\text{, provided } p \text{ is } \textbf{not} \text{ in the scope of a quantifier.} \qquad (PSL)$$

Translations

$(\exists x)A$ translates to $\neg(\forall x)\neg A$

$(\forall x|A : B)$ translates to $(\forall x)(A \Rightarrow B)$ (Range trading with $\forall$)

$(\exists x|A : B)$ translates to $(\exists x)(A \wedge B)$ (Range trading with $\exists$)

Useful facts from Predicate Calculus (proved in class—you may use them without proof):

We ***know*** that SLCS, WLUS (as well as GS-Leibniz "8.12(a)" and "8.12(b)") are **derived rules**. These are the following (I am using "GS"-notation for 8.12(a–b)):

$$\text{Same as PSL, without the condition: } A \equiv B \vdash C[p := A] \equiv C[p := B] \qquad (SLCS)$$

$$\text{if } \vdash A \equiv B\text{, then } \vdash C[p \setminus A] \equiv C[p \setminus B] \qquad (WLUS)$$

$$\text{if } \vdash A \equiv B\text{, then } \vdash \big(*\,x|C[p := A] : D\big) \equiv \big(*\,x|C[p := B] : D\big) \qquad (8.12(a))$$

$$\text{if } \vdash D \Rightarrow (A \equiv B)\text{, then } \vdash \big(*\,x|D : C[p := A]\big) \equiv \big(*\,x|D : C[p := B]\big) \qquad (8.12(b))$$

where in 8.12(a–b) "$*$" stands everywhere for the symbol "$\forall$", or the symbol "$\exists$".

► More "rules" and (meta)theorems. (Only the "$\forall$-versions" are listed. This should help you ***remember*** the "$\exists$-versions" that we also covered in class.):

(i)

$$\vdash A \equiv (\forall x)A\text{, } \textbf{provided } x \text{ is not free in } A$$

(ii) *Dummy renaming.*

$$\text{If } z \text{ does not occur in } (\forall x)A \text{ as either free or bound, then } \vdash (\forall x)A \equiv (\forall z)\big(A[x := z]\big)$$

(iii) $\forall$ *over* $\circ$ *distribution, where* $\circ$ *is "*$\vee$*" or "*$\Rightarrow$*".*

$$\vdash A \circ (\forall x)B \equiv (\forall x)(A \circ B)\text{, } \textbf{provided } x \text{ is not free in } A$$

(iv) $\forall$ *over* $\wedge$ *distribution.*

$$\vdash (\forall x)A \wedge (\forall x)B \equiv (\forall x)(A \wedge B)$$

(v) $\forall$ *commutativity (symmetry).*

$$\vdash (\forall x)(\forall y)A \equiv (\forall y)(\forall x)A$$

(vi) *Specialization. Follows from* **Ax2** *and MP.* $(\forall x)A \vdash A[x := t]$, for any term t.

(vii) *Generalization.* If $\Gamma \vdash A$ and if, moreover, the formulas in Γ have **no free** x **occurrences**, then also $\Gamma \vdash (\forall x)A$.

(viii) $\forall$ *Monotonicity.* If $\Gamma \vdash A \Rightarrow B$ so that the formulas in Γ have **no free** x **occurrences**, then we can infer

$$\Gamma \vdash (\forall x)A \Rightarrow (\forall x)B$$

(ix) $\forall$ *Introduction; a special case of* $\forall$ *Monotonicity that uses (i) above.* If $\Gamma \vdash A \Rightarrow B$ so that neither the formulas in Γ nor A have **any free** x **occurrences**, then we can infer

$$\Gamma \vdash A \Rightarrow (\forall x)B$$

(x) *The "$\forall\ \equiv$"-rule—a special case of WLUS.* If $\Gamma \vdash A \equiv B$ so that the formulas in Γ have **no free** x **occurrences**, then we can infer

$$\Gamma \vdash (\forall x)A \equiv (\forall x)B$$

(xi) Finally, the *Auxiliary Variable ("witness") Metatheorem.* If $\Gamma \vdash (\exists x)A$, and if y is a variable that ***does not*** occur as either free or bound variable in any of A or B or the formulas of Γ, then

$$\Gamma, A[x := y] \vdash B \text{ implies } \Gamma \vdash B$$

Semantics facts

Propositional Calculus	Predicate Calculus
(Boolean Soundness) $\vdash A$ implies $\models_{\text{taut}} A$	$\vdash A$ does **NOT** imply $\models_{\text{taut}} A$
(Post) $\models_{\text{taut}} A$ implies $\vdash A$	However, (**Ax1**) $\models_{\text{taut}} A$ implies $\vdash A$
	(Pred. Calc. Soundness) $\vdash A$ implies $\models A$
	(Gödel Completeness) $\models A$ implies $\vdash A$

CAUTION! The above facts/tools are only a fraction of what we have covered in class. They are ***very important and very useful***, and that is why I list them for your reference here.

You can still use *without proof* ***ALL*** the things we have covered (such as "one-point-theorem", "deMorgan's laws", $\exists$-versions of all the above, etc.).

But these—unlisted ones—are up to you to remember and to correctly state!

When in doubt of whether or not a "tool" you are about to use ***was*** *covered in class, then* ***prove*** *the validity/fitness of the tool before using it!*