Informal Semantics

This, our last set of notes, is on informal semantics.[†] In particular we are interested in using the *soundness theorem* of predicate calculus to construct "counterexamples".

Central to all this is the concept of interpretation. Until now we treated formulas as "meaningless strings of symbols" which we knew how to manipulate with our logical rules of inference.

But what do all these symbols mean?

1. Interpretations

We do an interpretation of a formula in the following steps:

(1) Choose a *non empty* set D from where the object variables take their values.

(2) Interpret and replace each *nonlogical symbol of the formula* by an appropriate "concrete" symbol. Namely,

  (a) A constant a will be interpreted/replaced by some concrete (fixed) value in D. We denote this by a^D.

  (b) A function f will be interpreted/replaced by a concrete function f^D of the same arity as f. f^D will take inputs from D and give outputs to D.

  (c) A predicate P will be interpreted/replaced by a concrete predicate P^D of the same arity as P. P^D will take inputs from D and give as outputs $\underline{t}$ or $\underline{f}$.

(3) Each formal occurrence of $(\forall x)$ and $(\exists x)$ in the formula is replaced (for emphasis) by $(\forall x \in D)$ and $(\exists x \in D)$ respectively.

(4) If we apply (1)–(3) to formula A, then we denote by A^D the concrete formula that we have obtain.

[†]Formal, so-called Tarski-semantics are treated in [2, 3]. For our course this present note is *all* you need.

After having chosen D there are many choices for symbol interpretations (infinitely many, if D is infinite). Thus, (the choice of) D does *not* uniquely determine A^D. We are comfortable that the context (in the examples that we consider later on) will fend against any ambiguity.

1.1 Example. Consider the formula $P(x, x)$, where P is a 2-ary predicate. Here are a few possible interpretations:

1. $D = \mathbb{N}$ (the natural numbers, $\{0, 1, 2, \ldots\}$), $P^D =<$.

   (by "$P^D =<$" I mean that P was interpreted/replaced by "$<$", the "less than" relation on numbers).

   Thus, $\big(P(x, x)\big)^D$ is this formula over $\mathbb{N}$: $\quad x < x$. By the way, *no* value of x makes this true ($\underline{t}$).

2. $D = \mathbb{N}$, $P^D =\leq$.

   Thus, $\big(P(x, x)\big)^D$ is this formula over $\mathbb{N}$: $\quad x \leq x$. By the way, *every* value of x makes this true ($\underline{t}$).

3. $D = \{0, \{0\}\}$, $P^D =\in$. ("$\in$" is the concrete "is a member of" relation of set theory).

   Thus, $\big(P(x, x)\big)^D$ is this formula over D: $\quad x \in x$.

   Note that *every* value of x makes this false ($\underline{f}$). Indeed, $0 \in 0$ is false because (the right copy of) 0 has no members—it is atomic, not a set—and $\{0\} \in \{0\}$ is false because (the right copy of) $\{0\}$ contains the element "0", *not* the element "$\{0\}$".

□

1.2 Example. Consider the formula $f(x) \approx f(y) \Rightarrow x \approx y$, where f is a 1-ary (unary) function. Here are a few possible interpretations:

1. $D = \mathbb{N}$, $f^D(x) = x + 1$ for all x.

   Thus, $\big(f(x) \approx f(y) \Rightarrow x \approx y\big)^D$ is this formula over $\mathbb{N}$:

   $$x + 1 = y + 1 \Rightarrow x = y$$

   By the way, *every* value of x and y makes this formula true ($\underline{t}$).

2. $D = \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers, $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. We take here $f^D(x) = x^2$ for all x.

   Thus, $\big(f(x) \approx f(y) \Rightarrow x \approx y\big)^D$ is this formula over $\mathbb{Z}$:

   $$x^2 = y^2 \Rightarrow x = y$$

   The above is true for some, and false for some other, values of x and y. For example, it is false for $x = -2$ and $y = 2$.

□

1.3 Example. Consider the formula

$$x \approx 0 \Rightarrow (\forall x)x \approx 0 \tag{1}$$

Here are a few possible interpretations:

1. $D = \{3\}$, $0^D = 3$.

   "0" is a nonlogical symbol in (1), and as such it has *NO* fixed meaning.

   So, I *must* interpret it as (replace it by) some element of D. Since there is only one such element my only option is to set "$0^D = 3$", as I did above.

   Thus, (1) is interpreted as the following formula over D:

   $$x = 3 \Rightarrow (\forall x \in D)x = 3 \tag{2}$$

   By the way, (2) is true for *all values of* x. The *only* value available to us to plug into the (***free***) x is "3". If I do so, then I am looking at the true formula

   $$3 = 3 \Rightarrow (\forall x \in D)x = 3$$

   (Why true? Because "$3 = 3$" is $\underline{t}$ and so is "$(\forall x \in D)x = 3$", since it says "all values x in D equal 3".)

2. $D = \{3, 5\}$, and again $0^D = 3$ (I could have taken $0^D = 5$ instead).

   Thus, (1) is interpreted as the following formula over this D:

   $$x = 3 \Rightarrow (\forall x \in D)x = 3 \tag{3}$$

   Now, (3) is **false** ($\underline{f}$) for $x = 3$, because "$3 = 3$" is $\underline{t}$ as before, but "$(\forall x \in D)x = 3$" is $\underline{f}$, since it still says "all values x in D equal 3", which now fails (D now has two elements. It is not true that both are equal to 3).

   Note that (3) is true ($\underline{t}$) for $x = 5$:

   $$5 = 3 \Rightarrow (\forall x \in D)x = 3$$

3. In class we took $D = \mathbb{N}$ and $0^D = 0$. This led to the interpretation

   $$x = 0 \Rightarrow (\forall x \in \mathbb{N})x = 0$$

   which is false for $x = 0$.

□

1.4 Example. Let us look at a couple of interpretations of

$$(\forall x)(x \in y \equiv x \in z) \Rightarrow y \approx z \tag{1}$$

1. If we take D to be the collection of all sets,† and $\in^D$ to mean "belongs to" (which we still denote as "$\in$") then we get

   $$(\forall x \in D)(x \in y \equiv x \in z) \Rightarrow y = z$$

   which is set theory's requirement ("axiom of extensionality") that two sets, y and z, are equal if they have the same elements. This is true for all the values (from D) of the free variables y and z.

2. Take $D = \mathbb{N}$ and $\in^D = <$. Then, (1) interprets into:

   $$(\forall x \in \mathbb{N})(x < y \equiv x < z) \Rightarrow y = z$$

   which is obviously true for all values (from $\mathbb{N}$) of the variables y, z.

3. Take $D = \mathbb{N}$ and $\in^D = \,|$, where by "$|$" we denote "divides" (with remainder 0). Then, (1) interprets into:

   $$(\forall x \in \mathbb{N})(x \,|\, y \equiv x \,|\, z) \Rightarrow y = z$$

   which is also obviously true for all values (from $\mathbb{N}$) of the variables y, z.

   However,

4. Take $D = \mathbb{Z}$ and $\in^D = \,|$. Now, (1) interprets into:

   $$(\forall x \in \mathbb{Z})(x \,|\, y \equiv x \,|\, z) \Rightarrow y = z$$

   which is **NOT** true for all values (from $\mathbb{Z}$) of the variables y, z.

   For example,

   $$(\forall x \in \mathbb{Z})(x \,|\, 2 \equiv x \,|\, -2) \Rightarrow 2 = -2$$

   is false, for the hypothesis $(\forall x \in \mathbb{Z})(x \,|\, 2 \equiv x \,|\, -2)$ is true (2 and -2 do have the same divisors), but the conclusion $2 = -2$ is false.

□

2. *Soundness*

We will now state a few definitions.

2.1 Definition. (Valid in an interpretation) A formula A is *valid in an interpretation with domain D*, in symbols

$$\models_D A$$

iff A^D (see item 4, p.1) is true (i.e., $\underline{t}$) ***for all the values—from*** D***—of the free variables***.

†**Small print.** We are **NOT** interested here in esoteric issues such as: "But this D is not a set".

2.2 Definition. (Universally (or Logically or Absolutely) Valid) A formula A is *universally valid* (also, "absolutely" or "logically" valid) ***iff*** $\models_D A$ for all possible choices of interpretations D.

We indicate that A is universally valid by

$$\models A$$

i.e., we drop the subscript from "$\models$".

Remember that we have (consciously) chosen an imperfect notation. When we say "for all possible choices of interpretations D" recall that—D being but one ingredient of the interpretation—we mean "for all choices of domain D and all choices of interpretations $a^D, \ldots,\ f^D, \ldots\ P^D, \ldots$ of all the logical symbols $a, \ldots, f, \ldots, P, \ldots$ that occur in the formula".

2.3 Remark. All axioms are universally valid. For example, $x \approx x$ interprets as "$x = x$" over any D, and is clearly true no matter what value (from D) we plug into x.

$(\forall x)A \Rightarrow A[x := t]$ is universally valid. A careful proof (based on a careful definition) is beyond our ambitions and goals here. Suffice it to say that when interpreted, this formula says that if A is true for all values of x then it is also true for any choice of a fixed value of x. And this sounds true! □

We state without proof.

2.4 Theorem. (Soundness) *If* $\vdash A$, *then* $\models A$.

Caution! We are in predicate calculus. Thus, the above holds, but "if $\vdash A$, then $\models_{\text{taut}} A$" is false. For example, $\vdash x \approx x$, (and also $\models x \approx x$) but

$$\not\models_{\text{taut}} x \approx x$$

2.5 Theorem. (Gödel's Completeness Theorem) *If* $\models A$, *then* $\vdash A$.

We will never need the completeness theorem in this course. It is only stated here for the sake of completeness (no pun intended).

Soundness, just as in the case of Propositional Calculus, serves the purpose of obtaining counterexamples. Thus, if for some formula A we do *not* believe that $\vdash A$, we only need to show that $\not\models A$.

2.6 Example. Question: Can our logic derive the "rule"

$$\text{If } \Gamma \vdash A, \text{ then } \Gamma \vdash (\forall x)A, \textit{ without a condition on } x\textit{?}$$

Well, if it could derive the above, it could also derive *strong generalization* below, by setting $\Gamma = \{A\}$.[†]

$$A \vdash (\forall x)A \tag{1}$$

Why not? Because (DThm) (1) would yield

$$\vdash A \Rightarrow (\forall x)A \tag{2}$$

By soundness, (2) yields

$$\models A \Rightarrow (\forall x)A \tag{3}$$

for no matter what A stands for. (3) is no good, as we saw in Example 1.3—choosing A to be $x \approx 0$ yields bad news—so (2) is no good, so (1) does not hold in our logic. □

2.7 Example. Knowing that strong generalization is illegal in our logic[‡] we can show that certain other suggested "rules" are impossible (*underivable*) by "reducing them to strong generalization", that is by saying

If I could have this rule, then I could do strong generalization. Impossible.

For example, here is why we cannot derive

$$A \equiv B \vdash (\forall x)(C[p := A] \Rightarrow D) \equiv (C[p := B] \Rightarrow D) \tag{1}$$

Take B to be the formula *true*, C to be just $\neg p$, and D be the formula *false*. Then (1) yields

$$A \vdash (\forall x)A \tag{2}$$

But (2) is unacceptable (by the previous example), hence so is (1).

[†] Then $A \vdash A$, hence $A \vdash (\forall x)A$.

[‡] We *have to* say "in our logic", which is that of [1, 2]. In the logic of [3], $A \vdash (\forall x)A$ is perfectly "legal".

In slow motion, we have (keeping an eye on what we said B, C, D are)

$$
\begin{aligned}
& (\forall x)A \\
= & \left\langle \text{WLUS and } \models_{\text{taut}} X \equiv \neg X \Rightarrow \textit{false}.\ C[p := A] \text{ is the string } \neg A \right\rangle \\
& (\forall x)(C[p := A] \Rightarrow D) \\
= & \left\langle (1) \text{ and Red. true, since } A \vdash A \equiv \textit{true} \right\rangle \\
& (\forall x)(C[p := B] \Rightarrow D) \\
= & \left\langle \text{Recall what } B, C, D \text{ are, and use } \models_{\text{taut}} X \equiv X \right\rangle \\
& (\forall x)(\neg \textit{true} \Rightarrow \textit{false}) \\
= & \left\langle \text{drop } \forall \text{, by } \vdash X \equiv (\forall y)X \text{ when } X \text{ has no free } y \right\rangle \\
& \neg \textit{true} \Rightarrow \textit{false}
\end{aligned}
$$

The last formula is a tautology, hence a theorem. Thus, the first line is a theorem. **Assumption used was A.**

In the same manner one shows that "8.12(b)" ((1) is "8.12(a)") is *not* strong, i.e., the following is *not* correct.

$$D \Rightarrow (A \equiv B) \vdash (\forall x)(D \Rightarrow C[p := A]) \equiv (D \Rightarrow C[p := B])$$

(see [2]). □

2.8 Example. Why insist on choosing a non empty D ?

Take any formula A. Clearly $(\forall x)A \Rightarrow (\exists x)A$ is false when interpreted on an *empty domain* D.

Why? "$(\forall x)A$" is true, since there are *no* x values in D to use towards a counterexample. On the other hand, "$(\exists x)A$" is false, for it says "there exists an x value that verifies A". However, there are *no* values in D to choose from.

"Big deal" you say. Why should we worry about that?

Because it also happens that

$$\vdash (\forall x)A \Rightarrow (\exists x)A \tag{1}$$

If we allow empty D, the above argument shows

$$\not\models (\forall x)A \Rightarrow (\exists x)A$$

contradicting soundness, something we do not want to allow! □

2.9 Exercise. Prove (1).

Bibliography

[1] David Gries and Fred B. Schneider. *A Logical Approach to Discrete Math*. Springer-Verlag, New York, 1994.

[2] G. Tourlakis. A basic formal equational predicate logic. Technical Report CS-1998-09, York University, Dept. of Comp. Sci., 1998.

[3] G. Tourlakis. On the soundness and completeness of equational predicate logics. Technical Report CS-1998-08, York University, Dept. of Comp. Sci., 1998.