Chapter 1

Intro. to Logic for Computer Science

What led to formal logic?

Cantor's *naïve* (same as *informal* and non axiomatic[†]) Set Theory did, as it was plagued by *paradoxes* the most famous of which (and least "technical") being pointed out by Bertrand Russell and thus nicknamed "Russell's paradox".[‡]

Thus the world of mathematics witnessed both a rapid development of *formal* logic (i.e., logic that is based on *form* or *syntax* [of mathematical statements and proofs]) and at the same time led to major mathematical works that attempted to demonstrate that set theory, and indeed *all mathematics* can be based on formal logic, rather than on informal and sloppy arguments that lead to trouble.

Such pioneering works were Russell and Whitehead's *Principia Mathematica* [WR12] and Bourbaki's long sequence of volumes titled *Éléments de Mathématique*, which begins with the volume [Bou66].

1.1. Russell's "paradox"

Cantor's informal (= non axiomatic, *and* sloppily argued semantically) Set Theory is the theory of collections (i.e., sets) of objects.

This theory studies operations on sets, properties of sets, and aims to use set theory as the foundation of all mathematics. Naturally, mathematicians "do" set theory of *mathematical object collections*. We have learnt some elementary aspects of set theory at high school and in MATH 1190 and/or EECS 1019.

1. **Notation**. *Sets given by listing.* For example, $\{1, 2\}$ is a set that contains precisely the objects 1 and 2, while $\{1, \{5, 6\}\}$ is a set that contains pre-

[†]Euclid's Geometry was axiomatic, but was based on informal reasoning. Formal logic was nowhere in existence in his time.

[‡]From the Greek word "paradoxo", meaning against one's belief; a contradiction.

cisely the objects 1 and $\{5,6\}$. The braces $\{$ and $\}$ are used to show the collection/set by outright listing.

2. **Notation**. *Sets given by "defining property"*. But what if we cannot (or will not) explicitly list all the members of a set? Then we may define what objects x get in the set/collection by having them to pass an entrance requirement, $P(x)$. An object x gets in the set iff (*if and only if*) $P(x)$ is true of said object.

   We denote the collection defined by the entrance condition $P(x)$ by

   $$\{x : P(x)\} \tag{1}$$

   reading it "the set of all x *such that* (this is ":") $P(x)$ is true [or holds]"

3. "$x \in A$" is the assertion that "object x is in the set A". Of course, this assertion may be true or false or "it depends", just like the assertions of algebra $2 = 2$, $3 = 2$ and $x = y$ are so (respectively).

4. $x \notin A$ is the negation of the assertion $x \in A$.

5. **Properties**

   - Sets are *named* by letters of the Latin alphabet. Naming is pervasive in mathematics as in, e.g., "let $x = 5$" in algebra.

     So we can write "let $A = \{1,2\}$" and let "$c = \{1,\{5,6\}\}$" to give the names A and c to the two example sets above, ostensibly because we are going to discuss these sets, and refer to them often, and it is cumbersome to keep writing things like $\{1,\{5,6\}\}$. Names are *not permanent*;[†] they are *local* to a discussion (argument).

   - Equality of sets (repetition and permutation do not matter!)

     Two sets A and B are equal iff they have the same members. Thus order and multiplicity do not matter! E.g., $\{1\} = \{1,1,1\}$, $\{1,2,1\} = \{2,1,1,1,1,2\}$.

   - The fundamental equivalence pertaining to definition of sets by "defining property": So, if we name the set in (1) above, S, that is, if we say "let $S = \{x : P(x)\}$", then "$x \in S$ iff $P(x)$ is true"

   By the way, we almost never say "is true" unless we want to shout out this fact. We would say instead: "$x \in S$ iff $P(x)$".

Let us pursue, as Russell did, the point made in the last bullet above. Take $P(x)$ to be specifically the assertion $x \notin x$. He then gave a name to

$$\{x : x \notin x\}$$

[†]OK, there *are* exceptions: $\emptyset$ is the permanent name for the *empty set* —the set with no elements at all— and for that set only; $\mathbb{N}$ is the permanent name of the set of all *natural numbers*.

say, R. But then, by the last bullet above,

$$x \in R \text{ iff } x \notin x \tag{2}$$

If we now *believe*,[†] as *Cantor*, the father of set theory *asserted*, that every $P(x)$ defines a set, *then R is a set*.

What is wrong with that?

Well, if R is a set then this object has the proper *type* to be plugged into the *variable of type set*, namely, x, throughout the equivalence (2) above. But this yields the contradiction

$$R \in R \text{ iff } R \notin R \tag{3}$$

This contradiction is called the Russell's Paradox.

This and similar paradoxes motivated mathematicians to develop formal symbolic logic and look to axiomatic set theory[‡] as a means to avoid paradoxes like the above.

What is "formal" logic? It is logic based strictly on form (syntax) and rigid purely syntactic rules on how proofs are to be formed (written). This is what we learn to practise in MATH 1090.

[†] Informal mathematics often relies on "I know so" or "I believe" or "it is obviously true".

[‡] There are many flavours or axiomatisations of set theory, the most frequently used being the "ZF" set theory, due to Zermelo and Fraenkel.

Bibliography

[Bou66] N. Bourbaki, *Éléments de Mathématique; Théorie des Ensembles*, Hermann, Paris, 1966.

[WR12] A. N. Whitehead and B. Russell, *Principia mathematica*, vol. 2, Cambridge University Press, Cambridge, 1912.