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MATH1090 A. Problem Set No 4 —SOLUTIONS

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In what follows, if I say "give a proof of $\vdash A$" or "show $\vdash A$" this means to give an Equational or Hilbert-style proof of A, unless some other proof style is required (e.g., Resolution).

Annotation is always required! Never-ever omit the "$\Leftrightarrow$" from an Equational proof!

1. (5 MARKS) Prove using 1st-Order **Soundness** (**Required**):

$$\nvdash \Big((\forall \mathbf{x})A \to (\forall \mathbf{x})B\Big) \to (\forall \mathbf{x})(A \to B) \tag{1}$$

Proof. To show that the wff in (1) is *NOT* a theorem I will show that the wff following "$\nvdash$" is **NOT valid**.

Practically, to prove

$$\not\models \Big((\forall \mathbf{x})A \to (\forall \mathbf{x})B\Big) \to (\forall \mathbf{x})(A \to B) \tag{2}$$

we do so for some simple wff A and B that we carefully choose in a *familiar* part of mathematics. So, if (2) is not valid for the special case then it is not valid, *period!*

As recommended in class/Notes I will take some appropriate ATOMIC special cases of A and B in an appropriate interpretation $\mathfrak{D} = (\mathbb{N}, \mathcal{M})$ where the interpretation is over $\mathbb{N}$ and then show that the wff over $\mathbb{N}$ (in (2′) below) is $\mathbf{f}$.

The atomic choices (relative to $\mathbb{N}$) are "$x > 0$" for A and "$x > 5$" for B. Thus the **wff-part** in (2) interprets as

$$\underbrace{\Big(\overbrace{(\forall x \in \mathbb{N})x > 0}^{\mathbf{f}} \to \overbrace{(\forall x \in \mathbb{N})x > 5}^{\mathbf{f}}\Big)}_{\mathbf{t}} \to \overbrace{(\forall x \in \mathbb{N})(x > 0 \to x > 5)}^{\mathbf{f}} \qquad (2')$$

Thus $(2')$ is **false** and therefore (2) is established. (1) is proved.

It is a *significant error* (**pushes the mark to zero**) to take "A" in our interpretation one that has no free x.

Why? Because then $\vdash A \equiv (\forall x)A$ and we **know** that $\vdash (A \to (\forall x)B) \equiv (\forall x)(A \to B)$.

$\square$

2. (5 MARKS) *Prove* that *IF we have*

$$\vdash (\exists \mathbf{x})A \to A[\mathbf{x} := z] \tag{1}$$

($\mathbf{z}$ fresh), THEN we also have

$$\vdash (\exists \mathbf{x})A \to (\forall \mathbf{x})A \tag{2}$$

Now also answer these three subsidiary questions:

(a) (2 MARKS) What does (2) say in words?

(b) (2 MARKS) Can you find a very simple specific example of a wff "A" over the natural numbers that makes (2) a non-theorem?

Prove that the wff you proposed *IS* a non theorem!

(c) (2 MARKS) What can you conclude from (b) about the validity of (1)?

Proof. Given (1). Since $(\exists \mathbf{x})A$ has no free $\mathbf{z}$ (it is chosen to be fresh!) I obtain the following by $\forall$-Intro:

$$\vdash (\exists \mathbf{x})A \to (\forall \mathbf{z})A[\mathbf{x} := z] \tag{1'}$$

I now have the following short proof by the "variant theorem" (or, "bound var. renaming"):

$$
\begin{aligned}
&\quad (\exists \mathbf{x})A \to (\forall \mathbf{z})A[\mathbf{x} := z]\\
&\Leftrightarrow \langle \text{WL + bound var. renaming; Denom: } (\exists \mathbf{x})A \to \mathbf{p} \rangle\\
&\quad (\exists \mathbf{x})A \to (\forall \mathbf{x})A[\mathbf{x}]
\end{aligned}
$$

The above Equational proof and (1′) together establish (2).

Now the subsidiary questions:

(a) (2) says "if $A(\mathbf{x})$ is true for SOME $\mathbf{x}$-value, then it is true for ALL $\mathbf{x}$-values!!!" (This cannot be possibly right!)

(b) Take $x = 0$ (over $\mathbb{N}$) for "A". Then (2) says:

$$\vdash (\exists x)x = 0 \to (\forall x)x = 0 \tag{2'}$$

NOTE that

$$\underbrace{\overbrace{(\exists x \in \mathbb{N})x = 0}^{\mathbf{t}} \rightarrow \overbrace{(\forall x \in \mathbb{N})x = 0}^{\mathbf{f}}}_{\mathbf{f}} \tag{$2''$}$$

By 1st-Order Soundness, $(2')$ is an invalid statement (because theoremhood of the wff in $(2')$ requires that $(2'')$ is true).

So (2) is not a theorem *schema* EITHER because we have shown a **special instance** —$(2')$— which is not.

(c) IF (1) is valid (that is, it truthfully IS a theorem) then —as we proved— so is (2).

We also proved that (2) is **NOT** a theorem; just now, in (b) above. This contradiction invalidates (1):

So (1) does NOT state a theorem!

□

3. (5 MARKS) *Use the ∃ elimination technique* —***Required***— *to show, for any A and B,*

$$\vdash (\exists \mathbf{x})(A \equiv \neg A) \to B$$

Proof. □

By DThm it suffices to prove

$$(\exists \mathbf{x})(A \equiv \neg A) \vdash B \tag{1}$$

instead.

1)	$(\exists \mathbf{x})(A \equiv \neg A)$	$\langle$hyp$\rangle$
2)	$A[\mathbf{x} := \mathbf{z}] \equiv \neg A[\mathbf{x} := \mathbf{z}]$	$\langle$**aux.** hyp for 1; $\mathbf{z}$ fresh for $(\exists \mathbf{x})A, B\rangle$
3)	B	$\langle 2 + \text{Post}\rangle$

(1) does *NOT* tautologically imply (3)! The wff in (1) is Prime, so I can make it **t** or **f** as I please. It is *NOT* therefore —Boolean-speaking— unsatisfiable (meaning "NOT always false")! So, I cannot conclude that (1) tautologically implies (3).

I can DO so with (2) as lhs of $\models_{taut}$, because (2) IS unsatisfiable!

4.

(4 MARKS) Prove

$$\vdash (\forall x)(\forall y)x = y \to (\forall y)y = y \tag{1}$$

Proof.

1)	$(\forall y)y = y$	$\langle$**Ax5** (*partial* Gen. of)$\rangle$
2)	$(\forall x)(\forall y)x = y \to (\forall y)y = y$	$\langle$1 + Post$\rangle$

□

(1 MARK) Also *explain precisely why* (1) is NOT an *instance* of **Ax2**.

Answer. The lhs (left of "$\to$", that is) in the wff of (1) is

$$(\forall x)\overbrace{\Big((\forall y)x = y\Big)}^{A} \tag{2}$$

Using (2) as the lhs of **Ax2** needs **THE** rhs

$$A[x := y] \tag{3}$$

in order to obtain $(\forall y)y = y$. But this substitution is illegal and aborts because y is *captured* by $(\forall y)$ if we insist to go ahead with it. □