York University

Faculties of Pure and Applied Science, Arts, Atkinson

MATH 2090. Problem Set #3. Posted October 19 , 2003

Due in the Course Box. November 6, 2003

Section A

In your proofs (formal or not) it is imperative to clearly state what **tools** you use (e.g., WLUS, sWLUS, MP, Leibniz, Monotonicity, Deduction Theorem, Generalization, Auxiliary Variable metatheorem, which axiom(s), etc.)

1. (5 Marks) Let $F = \emptyset$. Calculate (give the simplest possible answer).

   (1) $\bigcup F$

   (2) $\bigcap F$

   In problems 2–7 you may apply *informal proofs*, however, the only facts you may use are Logic, the Peano axioms, and whatever we proved from those in class [by the time of me writing this we have only proved $0 \le x$ and $x < y < z \Rightarrow x < z$ in class]. So, do **Not** assume in your proofs that you "know" anything about $\mathbb{N}$ that we have not proved in class **from the axioms**.

2. (5 Marks) Grade the following wrong "proof" by contradiction of $PA \vdash \neg Sx = x$. You *must* locate all errors *precisely*.

   Proof. We argue by contradiction, so let as *assume* the opposite of the required conclusion:

   $$Sx = x \tag{1}$$

   Now this yields $S0 = 0$ by doing specialisation on the generalisation—$(\forall x)Sx = x$—of (1). But this contradicts the specialisation $\neg S0 = 0$ that we obtain from the first Peano axiom $(\forall x)\neg Sx = 0$. $\square$

3. (5 Marks) OK, now give a correct inductive proof of $PA \vdash \neg Sx = x$.

4. (5 Marks) Here is another incorrect "proof" I want you to grade carefully and find *all* logical errors. Prove $PA \vdash \neg x < x$.

   Proof. We do induction.

   Basis: Verify $\neg 0 < 0$. That follows from the first $<$-axiom, that is, $(\forall x)\neg x < 0$, by specialisation.

   We now add the I.H.
$$\neg x < x \tag{2}$$
   and "goto" proving
$$\neg Sx < Sx \tag{3}$$
   Now, (3) equivales to
$$\neg(Sx = x \lor Sx < x) \tag{4}$$
   by the 2nd $<$-axiom, so I need to prove that $Sx = x \lor Sx < x$ yields *false*.

   I argue by cases:

   Case 1. If I have $Sx = x$, then, in particular I have $S0 = 0$ (specialisation from $(\forall x)Sx = x$) which contradicts $\neg S0 = 0$ obtained from the first axiom for "S" ($(\forall x)\neg Sx = 0$).

   Case 2. If I have $Sx < x$, then, in particular I have $S0 < 0$ which contradicts the first $<$-axiom: $(\forall x)x < 0$ (just specialise $[x := S0]$). □

5. (5 Marks) OK, now give a correct inductive proof of $PA \vdash \neg x < x$.

6. (5 Marks) Prove from the Peano axioms,
$$\text{PA} \vdash (\forall x, y, z)(x + y) + z = x + (y + z)$$
   NB. "$(\forall x, y, z)$" is a lazy way to write "$(\forall x)(\forall y)(\forall z)$".

   Hint. Do (simple) induction on z.

7. (5 Marks) Prove from Peano axioms
$$\text{PA} \vdash (\forall x)0 + x = x$$
   Hint. **Caution!** We do *not* have commutativity **from class or anywhere else, so do not use it**! Do (simple) induction on x.

Below you are also asked to do informal proofs, but you are allowed to "know a lot" about the natural numbers. Your tools will be logic, your knowledge, and Induction (simple or CVI, as the case demands).

You are not to use the Peano axioms (except for Induction). Of course, you are responsible to use correct principles, and, as always, please do not perform any leaps of logic/faith.

8. (5 Marks) Recall from class that GS notation "$(\sum i|0 \le i \le n : f(i))$" means $\sum_{i<n} f(i)$, which in turn denotes a function $g(n)$ of the variable n over $\mathbb{N}$ that is given inductively by

$$
\begin{aligned}
g(0) &= f(0) \\
g(n+1) &= g(n) + f(n+1)
\end{aligned}
$$

After this clarification/reminder do 12.4(d), p.242 of the GS text.

9. (10 Marks) Also do from the text p.244–246 informally:

$$\{12.15, 12.27\}$$

In problem 12.15—that is *required* to be done by induction—attach the *intuitive* meaning to the concept "number of elements of a finite set", just as we did at the end of our set theory notes (Part II). I.e., never mind the GS "axiom 11.12".

By the way, the symbol "$\#S$" in GS, where S is a set, is what we have denoted by "$|S|$" in class and in the notes.