# Chapter 1

# General Associativity for $\equiv, \vee$ and $\wedge$

This chapter retells (from the text, [Tou08]), hopefully more simply, the fact that Axiom Schema 1, namely,

$$\Big((A \equiv B) \equiv C\Big) \equiv \Big(A \equiv (B \equiv C)\Big) \tag{1}$$

implies that in a *chain of equivalences*

$$A_1 \equiv A_2 \equiv \cdots \equiv A_n \tag{2}$$

it is *unimportant how brackets are inserted*

This *unimportance* means that we claim the following theorem, (3), which we will prove here.

$$\vdash \Big(A_1, A_2, \cdots, A_n\Big) {\color{red}\equiv} \Big(A_1 \equiv \big(A_2 \equiv (A_3 \equiv \cdots \equiv A_n \cdots)\big)\Big) \tag{3}$$

where "$\Big(A_1, A_2, \cdots, A_n\Big)$" denotes **a fully parenthesised version of (2)** with **one** among the *many* possible bracketings being chosen, while the rhs of the red $\equiv$ denotes the formula with the *canonical* ***insertion*** *of brackets* (assuming all were missing), *from right to left*, as we learnt to do in class/text.

*And this is not all!* As the technique is the *same* for all three connectives $\equiv, \vee$ and $\wedge$, we in fact **prove that**

If $\circ$ is one of $\equiv, \vee$ and $\wedge$, then

$$\vdash \Big(A_1, A_2, \cdots, A_n\Big) {\color{red}\equiv} \Big(A_1 \circ \big(A_2 \circ (A_3 \circ \cdots \circ A_n {\color{red}\cdots^{*}})\big)\Big) \tag{GAssoc}$$

*The red $\cdots$ are all right brackets; right?

where "$\Big(A_1, A_2, \cdots, A_n\Big)$" denotes **a fully parenthesised version of (4) below** with **one** among the many possible bracketings being chosen, while the rhs of the red $\equiv$ in $(GAssoc)$ denotes the formula in (4), this time with the *canonical **insertion** of brackets* (assuming all were initially removed), *from right to left*, as we learnt to do in class/text.

$$A_1 \circ A_2 \circ \cdots \circ A_n \tag{4}$$

The (meta) proof is by induction on $n$. This meta proof has *formal* (Equational-style) proofs embedded in it. *Each $A_i$ is assumed to **NOT** be of the form* $B_1 \circ B_2 \circ \cdots$, that is, *all* the $\circ$ that "glue" the chain (4) together are shown in (4) —there are $n-1$ of them.

**Each $A_i$ is a wff, so in particular is fully parenthesised**.

**Proof. The proof hinges on the result**

$$\vdash \Big((A \circ B) \circ C\Big) \equiv \Big(A \circ (B \circ C)\Big) \tag{$Assoc$}$$

For $\circ$ being $\equiv$ or $\vee$, we have $(Assoc)$ by *Axiom Schemata 1 or 5* respectively. For $\circ$ being $\wedge$, we have $(Assoc)$ as *a **theorem***.

A good starting point for the induction is $n = 3$, since if $n = 1$ we have just $A_1$ and there is nothing to prove, and if we have $n = 2$ then we have $(A_1 \circ A_2)$ and thus there are no further brackets to insert, one way or another; $(A_1 \circ A_2)$ *is a wff as is.*

So, **basis**: $n = 3$. The claim is correct (that is, $(GAssoc)$ is indeed an *absolute theorem*) since the lhs of the red $\equiv$ is **one** of

- $\big(A_1 \circ (A_2 \circ A_3)\big)$ in which case we are done by the theorem (class/text) $\vdash X \equiv X$
- $\big((A_1 \circ A_2) \circ A_3\big)$ in which case we are done by the result $(Assoc)$ above

**Take as I.H. that, for all $k < n$, the claim** $(GAssoc)$ **on p.1 is true**, and prove the case for $n$ (**this is the I.S.**).

This "$n$" is **always** thought of as the **number of the $A_i$'s —not just the last subscript**. Thus, if we are looking, say, at $\Big(A_3, A_4, \cdots, A_n\Big)$ then we have $n-2$ $A_i$'s here, and **the I.H. applies!**

**Let's now do the I.S.** We have two cases:

**a)** **Case where** brackets in the lhs (of the red $\equiv$) in $(GAssoc)$ are inserted so that **the last glue applied** was the leftmost $\circ$, that is, like this

$$\Big(A_1 \circ \Big(A_2, \cdots, A_n\Big)\Big)$$

We now have a short Equational proof that settles the I.S. in this case on the I.H. that we have

$$\vdash \Big(A_2, \cdots, A_n\Big) \equiv \Big(A_2 \circ \big(A_3 \circ \cdots \circ A_n \cdots\big)\Big) \qquad (I.H. \text{ for Case a})$$

$$\begin{aligned}
&\Big(A_1 \circ \Big(A_2, \cdots, A_n\Big)\Big) \\
\Leftrightarrow\ &\langle Leib + \text{I.H. for Case a; Denom: } A_1 \circ \mathbf{p},\ \mathbf{p} \text{ fresh}\rangle \\
&\Big(A_1 \circ \Big(A_2 \circ \big(A_3 \circ \cdots \circ A_n \cdots\big)\Big)\Big)
\end{aligned}$$

**b)** **Case where** brackets in the lhs (of the red $\equiv$) in ($GAssoc$) are inserted so that the last glue applied was **NOT** the leftmost $\circ$, that is, the situation is like this:

$$\Big(\Big(A_1, A_2, \cdots A_k\Big) \circ \Big(A_{k+1}, \cdots, A_n\Big)\Big)$$

where $2 \le k < n$. **Why** $k < n$?

By the I.H. we have

$$\vdash \Big(A_1, A_2, \cdots A_k\Big) \equiv \Big(A_1 \circ \big(A_2 \circ \cdots \circ A_k \cdots\big)\Big) \qquad (I.H. \text{ for Case b})$$

We now have a short Equational proof that settles the I.S. in this case as well:

$$\begin{aligned}
&\Big(\Big(A_1, A_2, \cdots A_k\Big) \circ \Big(A_{k+1}, \cdots, A_n\Big)\Big) \\
\Leftrightarrow\ &\Big\langle Leib + \text{I.H. for Case b; Denom: } \mathbf{p} \circ \Big(A_{k+1}, \cdots, A_n\Big),\ \mathbf{p} \text{ fresh}\Big\rangle \\
&\Big(\Big(A_1 \circ \big(A_2 \circ \cdots \circ A_k \cdots\big)\Big) \circ \Big(A_{k+1}, \cdots, A_n\Big)\Big) \\
\Leftrightarrow\ &\langle Assoc, \text{ p.2}\rangle \\
&\Big(A_1 \circ \Big(\big(A_2 \circ \cdots \circ A_k \cdots\big) \circ \Big(A_{k+1}, \cdots, A_n\Big)\Big)\Big) \\
\Leftrightarrow\ &\langle \text{by Case b}\rangle \\
&\Big(A_1 \circ \Big(A_2 \circ \big(A_3 \circ \cdots \circ A_n \cdots\big)\Big)\Big)
\end{aligned}$$

$\square$

**1.0.1 Exercise.** Prove the theorem *Assoc* on p.2 for the case where $\circ$ is $\wedge$. $\square$

# Chapter 2

# General Commutativity for $\equiv, \vee$ and $\wedge$

This chapter retells (from the text) in a unified manner the fact that the **Commutativity Theorem Schema** below

$$\vdash A \circ B \equiv B \circ A \tag{$Comm$}$$

where $\circ$ is one of $\equiv, \vee$ or $\wedge$, implies that

$$\vdash A_1 \circ A_2 \circ \cdots \circ A_n \equiv A_{j_1} \circ A_{j_2} \circ A_{j_3} \circ \cdots \circ A_{j_n} \tag{$GComm$}$$

where the $\circ$-chain $A_{j_1} \circ A_{j_2} \circ A_{j_3} \circ \cdots \circ A_{j_n}$ is **any permutation** of the $\circ$-chain $A_1 \circ A_2 \circ \cdots \circ A_n$ that **we** may **choose**.

As for $(Comm)$, it is an **Axiom Schema** ((2) and (6) for the cases where $\circ$ is $\equiv$ or $\vee$ respectively)) or an **absolute theorem schema** (case where $\circ$ is $\wedge$).

Note that according to the results of Chapter 1, brackets need only be inserted if we need to achieve some visual effect.

**2.0.1 Lemma. (Swapping two End-Formulas (red))** $\vdash B \circ C \circ D \equiv D \circ C \circ B$.

*Proof.*

$$
\begin{aligned}
& B \circ C \circ D \\
\Leftrightarrow\ & \langle (Comm) \text{ above, pretending we inserted brackets around } B \circ C \rangle \\
& D \circ B \circ C \\
\Leftrightarrow\ & \langle (Comm) + (Leib);\ \text{Denom: } D \circ \mathbf{p} \rangle \\
& D \circ C \circ B
\end{aligned}
$$

**Note**: Of course, $C$ may be a long or short $\circ$-chain. □

**2.0.2 Metatheorem. (On swapping any two formulas (red) in a $\circ$-chain)** $\vdash A \circ B \circ C \circ D \circ E \equiv A \circ D \circ C \circ B \circ E$.

*Proof.*

$$
\begin{aligned}
& A \circ B \circ C \circ D \circ E \\
\Leftrightarrow\ & \langle (Leib) + \text{Lemma; Denom: } A \circ \mathbf{p} \circ E \rangle \\
& A \circ D \circ C \circ B \circ E
\end{aligned}
$$

**Note**: Of course, $A, C, E$ may be long or short $\circ$-chains. $\Box$

**2.0.3 Exercise.** Prove the theorem $(Comm)$ on p.5 for the case where $\circ$ is $\wedge$. $\Box$

# Bibliography

[Tou08] G. Tourlakis, *Mathematical Logic*, John Wiley & Sons, Hoboken, NJ, 2008.