

Lecture #6 (Continued; Sept. 25)

0.0.1 Definition. (Axioms of Boolean Logic) In the following, (1)–(11), A, B, C *name* or *stand for* arbitrary wff.

<u>Properties of \equiv</u>	
Associativity of \equiv	$((A \equiv B) \equiv C) \equiv (A \equiv (B \equiv C))$ (1)
Symmetry (commutativity) of \equiv	$(A \equiv B) \equiv (B \equiv A)$ (2)
<u>Properties of \perp, \top</u>	
\top and \perp	$\top \equiv \perp \equiv \perp$ (3)
<u>Properties of \neg</u>	
Introduction of \neg	$\neg A \equiv A \equiv \perp$ (4)
<u>Properties of \vee</u>	
Associativity of \vee	$(A \vee B) \vee C \equiv A \vee (B \vee C)$ (5)
Symmetry (commutativity) of \vee	$A \vee B \equiv B \vee A$ (6)
Idempotency of \vee	$A \vee A \equiv A$ (7)
Distributivity of \vee over \equiv	$A \vee (B \equiv C) \equiv A \vee B \equiv A \vee C$ (8)
“Excluded Middle”	$A \vee \neg A$ (9)
<u>Properties of \wedge</u>	
“Golden Rule”	$A \wedge B \equiv A \equiv B \equiv A \vee B$ (10)
<u>Properties of \rightarrow</u>	
Implication	$A \rightarrow B \equiv A \vee B \equiv B$ (11)

All of the above (1)–(11) except (3) are schemata for axioms. We call them *Axiom Schemata*, while (3) is an *Axiom*. Each axiom schema above defines *infinitely many axioms* that are its *Instances*.

So our axioms are (3) and all the instances of the Axiom Schemata (1), (2), (4)–(11).

We reserve the Greek letter Λ for the set of all Axioms of Boolean Logic. □

0.0.2 Definition. (Theorem Constructions —or, simply said, “Proofs”) Let Γ be some set of wff.

A *proof from Γ* is any *finite ordered* sequence of formulas that satisfy the following two *specifications*:

At every step of the Construction (Proof) we *may write*

Proof 1. *Any ONE formula* from Λ or Γ .

Proof 2. *Any wff A which is the result of an Application* of the rule *Leib* or rule *Eqn* to wff(s) that appeared in *THIS* proof *before* A.

A proof from Γ is also called “ Γ -proof”. □



0.0.3 Remark. (1) So, *a proof is a totally syntactic construct, totally devoid of semantic concepts.*

(2) Γ is a *convenient* set of “*additional hypotheses*”.

Syntactically the elements of Γ “behave” like the Axioms from Λ but *semantically they are NOT the same*:

While every member of Λ is a *tautology* by choice,

this *need NOT be the case for the members of Γ .*

(3) Since *every proof* (from some Γ) has *finite length*, only a finite part of Γ and Λ can ever appear in some proof. □ 

0.0.4 Definition. (Theorems) Any wff A that appears in a Γ -proof is called a Γ -theorem.

We also say, “ A is a theorem from Γ ”.

In symbols, the sentence “ A is a Γ -theorem”, is denoted by “ $\Gamma \vdash A$ ”.

If $\Gamma = \emptyset$ then we write $\vdash A$.



That is, Λ never appears to the left of the turnstile “ \vdash ”.



We call an A such that $\vdash A$ an absolute or logical theorem.

□

Lecture #7 (Sept. 30)



0.0.5 Remark. That A is a Γ -theorem is certified by a Γ -proof like this

$$B_1, \dots, B_n, A, C_1, \dots, C_m \tag{1}$$

the sequence (1) obeying the *specifications* of 0.0.2.

Clearly, the sequence (2) below also satisfies the specifications, since each specification for a B_i or A that utilises *rules* refers to formulas *to the left only*.

Thus the sequence (2) is also a Γ -proof of A !

$$B_1, \dots, B_n, A \tag{2}$$

The bottom line of this story is expressed as either

1. *If you are proving a theorem A , just stop as soon as you wrote it down with justification* in a proof!

OR

2. *A Γ -theorem is a wff that appears at the end of some proof.* □ 

Concatenating two Γ -proofs

$$A_1, \dots, A_n$$

and

$$B_1, B_2, \dots, B_r$$

results in a Γ -proof.

Indeed, checking

$$A_1, \dots, A_n, B_1, B_2, \dots, B_r$$

from left to right we give EXACTLY the same reasons we gave for writing the formulas down in each standalone proof.

The reader did not miss to note the similarity between a theorem construction and a formula construction.

Let us develop an *Inductive definition* for the concept “theorem” just as we did before for the concept “wff”.

So we learnt that a Γ -theorem, *let's call it A* , satisfies

1. *A is member of Λ or Γ*
2. *A appears in a Γ -proof as the result of an application of Eqn to wff to its left in the proof.*
3. *A appears in a Γ -proof as the result of an application of Leib to wff to its left in the proof.*

Let us rephrase remembering that a Γ -theorem is a formula that appears in a Γ -proof.

1. *A is member of Λ or Γ*
2. *A is the result of an application of Eqn to two Γ -theorems.*
3. *A is the result of an application of Leib to one Γ -theorem.*

0.0.6 Exercise. How do we do this?

By providing a Γ -proof where our target theorem appears, OR by using the Inductive Definition of the previous page.

- (1) $\{A\} \vdash A$, for any wff A
- (2) More generally, if $A \in \Sigma$, then $\Sigma \vdash A$
- (3) $\vdash B$, for all $B \in \Lambda$.

□

**0.0.7 Remark. (Hilbert-style Proofs)**

A Γ -proof is also called a “Hilbert-style proof” —in honour of the great mathematician *David Hilbert*, who was *the first big supporter* of the idea *to use logic as a TOOL in order to do mathematics*.

We arrange Hilbert proofs *vertically, one formula per line, numbered with its position number*, adding “*annotation*” *to the right of the formula we just wrote*, articulating briefly *HOW exactly* we followed the spec of Definition 0.0.2.

Practical Note. Forget numbering or annotation, or that *each line contains ONE wff* and as a result *forget a decent grade!* :) □

0.0.8 Example. (Some very simple Hilbert Proofs)

(a) We verify that “ $A, A \equiv B \vdash B$ ” (goes without saying, *for all wff A and B*).

Well, just write a proof of B with “ Γ ” being $\{A, A \equiv B\}$.


BTW, we indicate a finite “ Γ ” like $\{A, A \equiv B\}$ without the braces “ $\{ \}$ ” when writing it to the left of “ \vdash ”.

- (1) A \langle hypothesis \rangle
- (2) $A \equiv B$ \langle hypothesis \rangle
- (3) B $\langle(1) + (2) + (Eqn)\rangle$

 Incidentally, *members of Γ are annotated as “hypotheses” and going forward we just write “hyp”*.

Members of Λ we annotate as “Axioms”.



 Since A and B are arbitrary *undisclosed* wff, *the expression $A, A \equiv B \vdash B$ is a Theorem Schema* (a theorem, no matter what formulas we plug into A and B).



(b) Next verify the Theorem Schema

$$A \equiv B \vdash C[\mathbf{p} := A] \equiv C[\mathbf{p} := B]$$

Here you go:

- (1) $A \equiv B$ (hyp)
- (2) $C[\mathbf{p} := A] \equiv C[\mathbf{p} := B]$ ((1) + Leib)

(c) Something more substantial. *Our First Derived Rule!*

We establish the following *Theorem Schema* that we will refer to as *Transitivity* of \equiv —or simply “*Trans*”. **How?** *We write a Hilbert proof!*

$$A \equiv B, B \equiv C \vdash A \equiv C \quad (\text{Trans})$$

- (1) $A \equiv B$ $\langle \text{hyp} \rangle$
- (2) $B \equiv C$ $\langle \text{hyp} \rangle$
- (3) $(A \equiv B) \equiv (A \equiv C)$ $\langle (2) + (\text{Leib}), \text{Denom. “}A \equiv \mathbf{p}\text{” where } \mathbf{p} \text{ is “fresh”} \rangle$
- (4) $A \equiv C$ $\langle (1) + (3) + (\text{Eqn}) \rangle$

Why must \mathbf{p} be fresh?

Say A is $\mathbf{p} \wedge \mathbf{q}$. Then feeding B to \mathbf{p} I get $B \wedge \mathbf{q} \equiv B$

(d) *And a Tricky One!* Verify that “ $A \equiv A$ ” is an absolute theorem for all A . That is,

$$\vdash A \equiv A$$

No “HYP” in the proof below!!

- (1) $A \vee A \equiv A$ ⟨axiom⟩
- (2) $A \equiv A$ ⟨(1) + (Leib): $A[\mathbf{p} := A \vee A] \equiv A[\mathbf{p} := A]$ where \mathbf{p} is “fresh”⟩

□

0.0.9 Metatheorem. (Hypothesis Strengthening) If $\Gamma \vdash A$ and $\Gamma \subseteq \Delta$, then also $\Delta \vdash A$.

Proof. A Γ -proof for A is also a Δ -proof, since every time we say about a formula B in the proof “legitimate since $B \in \Gamma$ ” we can say instead “legitimate since $B \in \Delta$ ”. \square

0.0.10 Metatheorem. (Transitivity of \vdash) Assume $\Gamma \vdash B_1, \Gamma \vdash B_2, \dots, \Gamma \vdash B_n$. Let also $B_1, \dots, B_n \vdash A$. Then we have $\Gamma \vdash A$.

Proof.

We have Γ -proofs

$$\boxed{\dots, B_1} \tag{1}$$

$$\boxed{\dots, B_2} \tag{2}$$

\vdots

$$\boxed{\dots, B_n} \tag{n}$$

We also have a $\{B_1, \dots, B_n\}$ -proof

$$\boxed{\dots, B_i, \dots, A} \tag{n+1}$$

Concatenate all proofs (1)–(n) (in any order) and to the right of the result glue the proof (n+1). We have the following proof:

$$\boxed{\dots, B_1}, \boxed{\dots, B_2}, \dots, \boxed{\dots, B_n}, \boxed{\dots, B_i, \dots, A}$$

\uparrow
 obtained earlier; see box i

□