Lecture #10 —Continued (Oct. 9)

0.0.1 Metatheorem. (Hypothesis splitting/merging)
For any wff A, B, C *and hypotheses* Γ*, we have* $\Gamma \cup \{A, B\} \vdash C$ *iff* $\Gamma \cup \{A \wedge B\} \vdash C$*.*

Proof. (Hilbert-style)
(I) *ASSUME* $\Gamma \cup \{A, B\} \vdash C$ and *PROVE* $\Gamma \cup \{A \wedge B\} \vdash C$.

So, armed with Γ and $A \wedge B$ as *hypotheses* I have to prove C.

(1) $A \wedge B$ $\langle$hyp$\rangle$
(2) A $\langle$(1) + $A \wedge B \vdash A$ rule $\rangle$
(3) B $\langle$(1) + $A \wedge B \vdash B$ rule $\rangle$
(4) C $\langle$using HYP Γ + (2) and (3) $\rangle$

(II) *ASSUME* $\Gamma \cup \{A \wedge B\} \vdash C$ and *PROVE* $\Gamma \cup \{A, B\} \vdash C$.

Exercise, or see Text. □

0.0.2 Theorem. (Modus Ponens) $A, A \to B \vdash B$

Proof.

$$
\begin{aligned}
& A \to B \\
\Leftrightarrow\ & \langle \neg\vee\text{-theorem} \rangle \\
& \neg A \vee B \\
\Leftrightarrow\ & \langle (Leib) + \text{hyp } A + \text{Red-}\top\text{-META; “Denom:” } \neg \mathbf{p} \vee B \rangle \\
& \neg \top \vee B \\
\Leftrightarrow\ & \langle (Leib) + \text{theorem from class; “Denom:” } \mathbf{p} \vee B \rangle \\
& \bot \vee B \\
\Leftrightarrow\ & \langle \text{thm from class} \rangle \\
& B
\end{aligned}
$$

□

0.0.3 Theorem. (Cut Rule) $A \vee B, \neg A \vee C \vdash B \vee C$

Proof. We start with an AUXILIARY theorem —a Lemma— which makes the most complex hypothesis $\neg A \vee C$ usable (an EQUIVALENCE).

$$\begin{aligned} & \neg A \vee C \\ \Leftrightarrow\ & \langle \text{how to lose a NOT} \rangle \\ & A \vee C \equiv C \end{aligned}$$

Since $\neg A \vee C$ is a HYP hence also a THEOREM, the same is true for $A \vee C \equiv C$ from the Equational proof above.

$$\begin{aligned} & B \vee C \\ \Leftrightarrow\ & \langle (\textit{Leib}) + \text{Lemma; "Denom:" } B \vee \mathbf{p} \rangle \\ & B \vee (A \vee C) \\ \Leftrightarrow\ & \langle \text{shifting brackets to our advantage AND swapping wff} \rangle \\ & (A \vee B) \vee C \\ \Leftrightarrow\ & \langle (\textit{Leib}) + \text{HYP } A \vee B + \text{Red-}\top\text{-Meta; "Denom:" } \mathbf{p} \vee C \rangle \\ & \top \vee C \qquad \text{Bingo!} \end{aligned}$$

□

SPECIAL CASES of CUT:

0.0.4 Corollary. $A \vee B, \neg A \vee B \vdash B$

Proof. From 0.0.3 we get $A \vee B, \neg A \vee B \vdash B \vee B$.

We have also learnt the rule $B \vee B \vdash B$.

Apply this rule to the proof above that ends with "$B \vee B$" to get B. □

0.0.5 Corollary. $A \vee B, \neg A \vdash B$

Proof. Apply the rule $\neg A \vdash \neg A \vee B$.

We now can use the above Corollary! □

0.0.6 Corollary. $A, \neg A \vdash \bot$

Proof. Hilbert-style.

(1)	A	$\langle$hyp$\rangle$
(2)	$\neg A$	$\langle$hyp$\rangle$
(3)	$A \vee \bot$	$\langle 1$ + rule $X \vdash X \vee Y \rangle$
(4)	$\neg A \vee \bot$	$\langle 2$ + rule $X \vdash X \vee Y \rangle$
(5)	$\bot$	$\langle 3 + 4$ + rule 0.0.4$\rangle$

□

0.0.7 Corollary. (Transitivity of $\to$) $A \to B, B \to C \vdash A \to C$

Proof. (Hilbert style)

(1)	$A \to B$	$\langle$hyp$\rangle$
(2)	$B \to C$	$\langle$hyp$\rangle$
(3)	$A \to B \equiv \neg A \vee B$	$\langle \neg\vee$ thm$\rangle$
(4)	$B \to C \equiv \neg B \vee C$	$\langle \neg\vee$ thm$\rangle$
(5)	$\neg A \vee B$	$\langle (1, 3) + (Eqn) \rangle$
(6)	$\neg B \vee C$	$\langle (2, 4) + (Eqn) \rangle$
(7)	$\neg A \vee C$	$\langle (5, 6) + \text{CUT} \rangle$

□

The last line is provably equivalent to $A \to C$ by the $\neg\vee$ theorem.

0.0.8 Theorem. $A \to C, B \to D \vdash A \vee B \to C \vee D$

Proof. As in the proof of 0.0.3, the analysis of the two HYPs gives as the two theorems from our "Γ":

$$A \vee C \equiv C \tag{1}$$

and

$$B \vee D \equiv D \tag{2}$$

Thus,

$$\begin{aligned}
& A \vee B \to C \vee D \\
\Leftrightarrow\ & \langle \text{axiom} \rangle \\
& A \vee C \vee B \vee D \equiv C \vee D \\
\Leftrightarrow\ & \langle (\textit{Leib}) + (1);\ \text{“Denom:”}\ \mathbf{p} \vee B \vee D \equiv C \vee D \rangle \\
& C \vee B \vee D \equiv C \vee D \\
\Leftrightarrow\ & \langle (\textit{Leib}) + (2);\ \text{“Denom:”}\ C \vee \mathbf{p} \equiv C \vee D \rangle \\
& C \vee D \equiv C \vee D \qquad \text{Bingo!}
\end{aligned}$$

□

0.0.9 Corollary. (The RULE of Proof By Cases) $A \to C, B \to C \vdash A \vee B \to C$

Proof.

By 0.0.8, we get

$$A \to C, B \to C \vdash A \vee B \to C \vee C$$

But then

$$\begin{aligned} & A \vee B \to C \vee C \\ \Leftrightarrow & \langle \text{idemp. axiom + Leib; "Denom": } A \vee B \to \mathbf{p} \rangle \\ & A \vee B \to C \end{aligned}$$

□