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MATH 2090. Problem Set #2 (Sample Solutions).
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Section A

Probl. 1. From the text p.213–215: Do the problems

(i) 11.7(c). Do $ST \vdash \{b, c\} = \{b, e\} \equiv c = e$.

Proof. The $\Leftarrow$ is by "replacing equals for equals in terms" theorem applied to terms (see ToolBox for statement of theorem, or see "All about Leibniz" to see a proof).

For the $\Rightarrow$ note that by Part I (2.7) and definition of pair (Part II, 1.7) we need to prove

$$ST \vdash (\forall x)(x = b \vee x = c \equiv x = b \vee x = e) \Rightarrow c = e$$

OK, use ded-thm.

(1) $(\forall x)(x = b \vee x = c \equiv x = b \vee x = e)$ $\Big($assume$\Big)$

(2) $c = b \vee c = c \equiv c = b \vee c = e$ $\Big($(1) and spec.$\Big)$

(3) $c = b \vee c = e$ $\Big($(2), $c = c$ and red. true$\Big)$

(4) $e = b \vee e = c \equiv e = b \vee e = e$ $\Big($(1) and spec.$\Big)$

(5) $e = b \vee e = c$ $\Big($(4), $e = e$ and red. true$\Big)$

(6) $(c = b \vee c = e) \wedge (e = b \vee e = c)$ $\Big($(3, 5) and taut. impl.$\Big)$

(7) $(c = b \wedge e = b) \vee (c = b \wedge e = c) \vee (c = e \wedge e = b) \vee (c = e \wedge e = c)$
$\Big($(6) and taut. impl.$\Big)$

Now (7) yields what we want by proof by cases:

Case A. $c = b \wedge e = b$: Conclude $c = e$ by transitivity of $=$.

Case B. $c = b \wedge e = c$: Conclude $c = e$ by taut. implication.*

*Symmetry of $=$ also used. This and transitivity are trivially provable from Ax5 and Ax6.

Case C. $c = e \wedge e = b$: Conclude $c = e$ by taut. implication.

Case D. $c = e \wedge e = c$: Conclude $c = e$ by taut. implication.

□

(ii) 11.12(a)

Proof. (Informal) So, assume (use ded-thm) $S \subseteq T$ and $U \subseteq V$ and also let $x \in S \cup U$. By def. of $\cup$, we have two cases:

Case 1. $x \in S$. Then $x \in T$ by assumption, and also $x \in T \cup V$ by def. of $\cup$.

Case 2. $x \in U$. Then $x \in V$ by assumption, and also $x \in T \cup V$ by def. of $\cup$.

(iii) 11.15: Prove $(\exists x)(x \in S \wedge x \notin T) \Rightarrow S \neq T$.

Proof. Prove the contrapositive: $S = T \Rightarrow \neg(\exists x)(x \in S \wedge x \notin T)$. That is, prove $S = T \Rightarrow (\forall x)\neg(x \in S \wedge x \notin T)$.

But that is $S = T \Rightarrow (\forall x)(x \in S \Rightarrow x \in T)$, i.e., a tautological consequence of the logical half of the Extensionality theorem. □

(iv) 11.18: Prove $S \in \mathscr{P}(S)$.

Proof. We want $S \in \{x|x \subseteq S\}$.

$$
\begin{aligned}
& S \in \{x|x \subseteq S\} \\
\equiv & \left\langle \text{by } \in\text{-elim. Note that the next line is } (x \subseteq S)[x := S] \right\rangle \\
& S \subseteq S
\end{aligned}
$$

The previous line is a theorem. □

For 11.7(c) and 11.18 in the list above, formal proofs are required.

Also do

Probl. 2. Prove **informally** $ST \vdash (\forall a, b, c, d)\big(\{a, \{a, b\}\} = \{c, \{c, d\}\} \Rightarrow a = c \wedge b = d\big)$.

To avoid an embarrassing situation I note that the above is **not** the same problem that I assigned last year. Do you see the difference?

Proof. Suffices to prove this without the $\forall$ (I can then introduce the $\forall$ by generalisation, since ST has closed axioms).

Assume then $\{a, \{a, b\}\} = \{c, \{c, d\}\}$.

That is,

$$(\forall x)(x = a \vee x = \{a, b\} \equiv x = c \vee x = \{c, d\})$$

By specialisation I get

$$c = a \vee c = \{a, b\} \equiv c = c \vee c = \{c, d\} \tag{1}$$

and

$$a = a \vee a = \{a, b\} \equiv a = c \vee a = \{c, d\} \tag{2}$$

Hence (see also proof on p.1)

$$c = a \vee c = \{a, b\} \tag{3}$$

and

$$a = c \vee a = \{c, d\} \tag{4}$$

(3) and (4) and taut. implication yield as on p.1

$$(c = a \wedge a = c) \vee (c = a \wedge a = \{c, d\}) \vee (c = \{a, b\} \wedge a = c) \vee (c = \{a, b\} \wedge a = \{c, d\}) \tag{5}$$

The disjunction (5) yields just

$$a = c \tag{6}$$

by cases (by the 1st disjunct), since each of the other three disjuncts yield *false* by foundation: The 2nd yields $c \in c$; the 3rd yields $a \in a$ and the 4th yields $a \in c \in a$.

(6) transforms our hypothesis to $\{a, \{a, b\}\} = \{a, \{a, d\}\}$. This by problem 11.7(c) yields $\{a, b\} = \{a, d\}$ and one more application of problem 11.7(c) yields $b = d$. Done! $\square$

Probl. 3. Give a **formal** proof of $ST \vdash S \subset T \Rightarrow (\exists x)(x \notin S)$.

Proof. Invoke the deduction theorem, and assume hypothesis (line (1) below).

(1) $(\forall x)(x \in S \Rightarrow x \in T) \wedge \neg(\forall x)(x \in S \equiv x \in T)$ $\Big($assume$\Big)$

(2) $(\forall x)(x \in S \Rightarrow x \in T)$ $\Big($(1) and Post's theorem$\Big)$

(3)	$\neg(\forall x)(x \in S \equiv x \in T)$	$\big($(1) and Post's theorem$\big)$
(4)	$(\exists x)(x \in S \equiv \neg x \in T)$	$\big($(3), Post and WLUS$\big)$
(5)	$z \in S \equiv \neg z \in T$	$\big($by (4): <u>assume</u>; z fresh$\big)$
(6)	$z \in S \Rightarrow z \in T$	$\big($(2) and specialisation$\big)$
(7)	$z \notin S$	$\big($(5,6) and Post$\big)$
(8)	$(\exists x)x \notin S$	$\big($(7) and $A \vdash \exists A$-rule$\big)$

$\square$

Probl. 4. Prove <u>without using the axiom of foundation</u> that $1 \neq \{1\}$ and $\emptyset \neq \{\emptyset\}$.

For $1 \neq \{1\}$ we argue as in GS, p.197: The lhs is type $\mathbb{N}$ (atom) and the rhs is of type SET. So they cannot be equal.[†]

As for $\emptyset \neq \{\emptyset\}$, recall from class that $ST \vdash \emptyset \neq T \equiv (\exists x)x \in T$.

Well, from $\emptyset \in \{\emptyset\}$ we get $(\exists x)x \in \{\emptyset\}$ by the $\exists$-rule ($A[t] \vdash (\exists x)A[x]$). Thus, $\emptyset \neq \{\emptyset\}$.

[†] This is *informal*, but common-sensically correct. The rigorous way to go about it is to have an axiom that says "$x \in y$ is false for any y of type atom"—in symbols, $\neg(\exists y)y \in x$ for any atom-type x. This axiom captures the "obvious truth" that atoms have no set structure, they cannot contain any members. An atom is not equal to $\emptyset$ however, for the latter has type SET.