Adaptive Filters

Konstantinos G. Derpanis
York University
kosta@cs.yorku.ca

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Following the work of Andersson [1], this note reviews a class of filters for which all rotated versions are spanned by a set of basis filters. These filters represent a class of steerable filters. A key difference between the steerable filters detailed below and those of Freeman and Adelson [2] is that the *steering constraint*[1] is not enforced.

Here we consider polar-separable amplitude spectra of the form,

$$F(\rho, \theta; \phi) = R(\rho)A(\theta; \phi) \tag{2}$$

where (ρ, θ) denote the polar parameterization of the frequency domain, $R(\rho)$ represents the radial frequency portion of the spectra and $A(\theta; \phi)$ represents the angular component parameterized by θ, rotated (steered) by ϕ.

The ability to represent the response of such filters over a continuous range based on interpolated responses follows directly from the trigonometric identity:

$$\cos(\theta - \phi) = \cos(\phi)\cos(\theta) + \sin(\phi)\sin(\theta). \tag{3}$$

This states that the functions $\{cos(\theta), sin(\theta)\}$ span the set of shifts ϕ (i.e., rotations in Cartesian frequency space).

Given that the angular component is periodic with period 2π, by the Fourier series [4] one can express the angular component $A(\theta; \phi)$ as,

$$A(\theta; \phi) = \sum_{n=0}^{N} a_n \cos(n(\theta - \phi)), \tag{4}$$

[1] The *steering constraint* states that a rotated version of a filter can be written as a linear combination of (fixed) rotated versions of the same filter, formally

$$f^{\theta}(x, y) = \sum_{j=1}^{M} k_j(\theta) f^{\theta_j}(x, y), \tag{1}$$

where f^{θ} represents $f(x, y)$ rotated through an angle θ about the origin.

or equivalently in complex notation as,

$$A(\theta;\phi) = \sum_{n=-N}^{N} a_n e^{in\phi}. \tag{5}$$

Plugging (4) into (2), we find that $F(\rho, \theta; \phi)$ can be expressed as a weighted sum of sinusoidal functions:

$$\begin{aligned} F(\rho,\theta;\phi) &= R(\rho)\sum_{n=0}^{N} a_n \cos(n(\theta-\phi)) \\ &= \sum_{n=0}^{N} a_n R(\rho)\cos(n(\theta-\phi)) \\ &= \sum_{n=0}^{N} a_n R(\rho)\Big(\cos(n\phi)\cos(n\theta) + \sin(n\phi)\sin(n\theta)\Big) \\ &= \sum_{n=0}^{N} a_n \cos(n\phi)\Big(R(\rho)\cos(n\theta)\Big) + a_n \sin(n\phi)\Big(R(\rho)\sin(n\theta)\Big), \end{aligned} \tag{6}$$

or in complex notation,

$$\begin{aligned} F(\rho,\theta;\phi) &= \sum_{n=-N}^{N} a_n e^{-in\phi}\Big(R(\rho)e^{in\theta}\Big) \\ &= \sum_{n=-N}^{N} a_n e^{-in\phi}\Big(B_n(\rho,\theta)\Big), \end{aligned} \tag{7}$$

where $B_n(\rho,\theta) = R(\rho)e^{in\theta}$ represent the basis filters. Since the argument of $B_n(\rho,\theta)$ is a harmonic function of order n, Andersson [1] refers to B_n as a *harmonic basis filter*. The basis set contains $2N+1$ real-valued functions[2] from the set $\{R(\rho)cos(n\theta)$, $R(\rho)\sin(n\theta)\}_{n=0,\ldots,N}$.

For an application of these adaptive filters in the context of image orientation estimation the reader is referred to [3].

References

[1] M. Andersson. *Controllable Multidimensional Filters in Low Level Computer Vision*. PhD thesis, Linköping University, Sweden, SE-581 83 Linköping, Sweden, September 1992.

[2] The reason for the basis set containing $2N+1$ functions as opposed to $2N+2$ is that at $N=0$ the component containing $sin(n\theta)$ vanishes.

[2] W.T. Freeman and E.H. Adelson. The design and use of steerable filters. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 13:891–906, 1991.

[3] L. Haglund and D.J. Fleet. Stable estimation of image orientation. *IEEE International Conference on Image Processing*, pages 68–72, 1994.

[4] H.J. Weaver. *Applications of Discrete and Continuous Fourier Analysis*. John Wiley and Sons, 1983.