Fourier Motion

Konstantinos G. Derpanis
kosta@cs.yorku.ca

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In this note we derive the Fourier spectrum of a one-dimensional scene undergoing constant velocity v_x. An early derivation of this motion phenomenon in the literature can be found in [6].

Let $I_s(x)$ represent a 1D static scene. The Fourier transform of the static image $I_s(x)$ is defined as follows,

$$\hat{I}_s(\omega_x) = \int I_s(x) e^{-i\omega_x x} dt \tag{1}$$

A scene $I(x,t)$ that is translating with a constant velocity v_x and zero acceleration can be described as follows,

$$I(x,t) = I_s(x - v_x t) \tag{2}$$

The Fourier transform of $I(x,t)$ symbolized as $\hat{I}(\omega_x, \omega_t)$ will now be derived,

$$\begin{aligned}
\hat{I}(\omega_x, \omega_t) &= \iint I(x,t) e^{-i(\omega_x x + \omega_t t)} dx dt && (3)\\
&= \iint I_s(x - v_x t) e^{-i(\omega_x x + \omega_t t)} dx dt && (4)\\
&= \iint I_s(x - v_x t) e^{-i\omega_x x} e^{-i\omega_t t} dx dt && (5)\\
&= \int \left(\int I_s(x - v_x t) e^{-i\omega_x x} dx \right) e^{-i\omega_t t} dt && (6)
\end{aligned}$$

Now applying the time shifting property of the Fourier transform [4] (i.e. $x(t - t_0) \stackrel{F}{\longleftrightarrow} e^{-i\omega_x t_0} \hat{x}(\omega_x)$) yields,

$$\begin{aligned}
\hat{I}(\omega_x, \omega_t) &= \int \left(\hat{I}_s(\omega_x) e^{-i\omega_x v_x t} \right) e^{-i\omega_t t} dt && (7)\\
&= \hat{I}_s(\omega_x) \int e^{-i\omega_x v_x t} e^{-i\omega_t t} dt && (8)\\
&= \hat{I}_s(\omega_x) \delta(\omega_t + v_x \omega_x) 2\pi && (9)
\end{aligned}$$

Geometrically, a scene undergoing constant velocity results in a shearing of its Fourier spectrum in the ω_t direction [6]. The result is an oriented spectral line through the origin with slope $-1/v_x$. Analogously, when considering two spatial dimensions, the Fourier spectrum of a two-dimensional scene undergoing constant velocity consists of a plane through the origin. This insight forms the basis for several motion estimation algorithms that estimate velocity by identifying the orientation of this plane [6, 1, 2, 3, 5].

References

[1] E.H. Adelson and J.R. Bergen. Spatiotemporal energy models for the perception of motion. *Journal of the Optical Society of America-A*, 2(2):284–299, 1985.

[2] D.J. Heeger. Optical flow from spatiotemporal filters. *International Journal of Computer Vision*, 1(4):279–302, January 1988.

[3] B. Jahne. Motion determination in space-time images. In *European Conference on Computer Vision*, pages 161–173, 1990.

[4] A.V. Oppenheim, A.S. Willsky, and Nawab S.H. *Signals and Systems*. Prentice Hall, Upper Saddle River, NJ, 1997.

[5] E.P. Simoncelli. *Distributed Analysis and Representation of Visual Motion*. PhD thesis, MIT, 1993.

[6] A.B. Watson and A.J. Ahumada. Model of human visual-motion sensing. *Journal of the Optical Society of America-A*, 2(2):322–342, 1985.