Gaussian Integrals

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In this report, the area under the Gaussian curve,

$$G(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/(2\sigma^2)} \tag{1}$$

where σ is a constant controlling the spread of the Gaussian (see Fig. 1), is verified to be equal to 1, formally,

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/(2\sigma^2)} dx = 1 \tag{2}$$

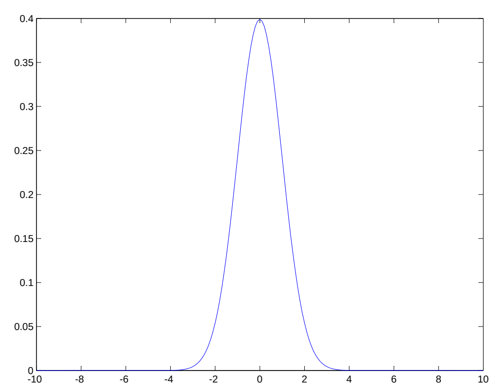

Figure 1: Gaussian curve with $\sigma = 1$.

The verification of this result can be obtained as follows. Denoting the area under the Gaussian as A,

$$\begin{aligned} A^2 &= \left(\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/(2\sigma^2)} dx \right)^2 & (3) \\ &= \left(\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/(2\sigma^2)} dx \right) \left(\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/(2\sigma^2)} dy \right) & (4) \end{aligned}$$

The product of two integrals can be expressed as a double integral, as follows,

$$A^2 = \frac{1}{\sigma^2 2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} e^{-y^2/(2\sigma^2)} dxdy \tag{5}$$

$$= \frac{1}{\sigma^2 2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/(2\sigma^2)} dxdy \tag{6}$$

Next, reexpress the area of integration in polar coordinates,

$$x = rcos\theta \tag{7}$$

$$y = rsin\theta \tag{8}$$

$$dxdy = rdrd\theta \tag{9}$$

the double integral becomes,

$$A^2 = \frac{1}{\sigma^2 2\pi} \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2/(2\sigma^2)} rdrd\theta \tag{10}$$

To solve the inner most integral, (i.e, $\int_0^\infty (\cdot) dr$) apply the substitution $u = r^2/(2\sigma^2), du = r/(\sigma^2)dr$,

$$A^2 = \frac{1}{\sigma^2 2\pi} \int_{0}^{2\pi} \left(\int_{0}^{\infty} e^{-u} \sigma^2 du \right) d\theta \tag{11}$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \left(\int_{0}^{\infty} e^{-u} du \right) d\theta \tag{12}$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} (1) d\theta \tag{13}$$

$$= \frac{1}{2\pi} (2\pi) \tag{14}$$

$$= 1 \tag{15}$$

Therefore, $A = 1$, proving result in Eq. 2.