Implementing Continuous Convolutions

Konstantinos G. Derpanis
Department of Computer Science and Engineering
York University
kosta@cs.yorku.ca

August 15, 2006.

The convolution operator plays an important role in image processing. In many cases, convolutions[1] are used in continuous domain models, for example, Gaussian convolutions used in scale-space (Koenderink, 1984). Given that image data is inherently discrete, a common approach to approximate the continuous model is to convert the continuous convolutions to their discrete counterparts by replacing the the integrals with sums. This however may lead to inaccuracies (Hummel & Lowe, 1989; van den Boomgaard, 2001).

Alternatively, Hummel and Lowe (Hummel & Lowe, 1989) and later van den Boomgaard and van der Weij (van den Boomgaard, 2001) propose the following scheme: (1) reconstruct the continuous form of the given discrete signal, (2) apply the continuous convolution, and (3) sample the result. Equivalently, this scheme can be reduced to a discrete convolution of the input signal in the discrete domain by the sampled version of the result of convolving the reconstruction kernel (i.e., interpolation) with the filter (computed offline). The remainder of this note formalizes this scheme.

Claim. The continuous convolution $f * w$ may be approximated by the discrete convolution, $F \circledast W_\phi$, where F is the sampled version of f, $S(f)$, ϕ denotes an

[1] In the continuous domain, convolution between two continuous signals, $f(x,y)$ and $g(x,y)$,is defined as:

$$h(x,y) = f * g = \int_{\mathbb{R}^2} f(x',y')g(x-x',y-y')dx'dy'. \tag{1}$$

In the discrete domain, convolution between two discrete signals, $f[x,y]$ and $g[x,y]$, is defined as,

$$h[x,y] = f \circledast g = \sum_{\mathbb{Z}^2} f[x',y']g[x-x',y-y']. \tag{2}$$

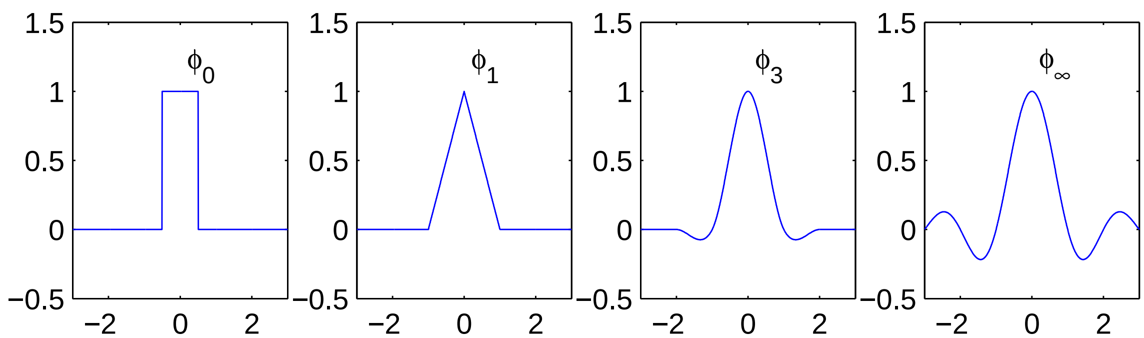

Figure 1: Several typical interpolation kernels (left-to-right): *nearest neighbour*, *bilinear*, *bicubic* and *sinc*.

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interpolation kernel (see Fig. 2 for example kernels) used to reconstruct a continuous approximation of f from the discrete signal, F, and W_ϕ denotes the convolution result of w with the interpolation kernel:

$$S(f * w) \approx S(f) \circledast S(w * \phi) = F \circledast W_\phi, \tag{3}$$

$S(\cdot)$ denotes the uniform sampling operator.

Proof. Let $I(F)$ denote the continuous reconstruction of discrete signal F with kernel ϕ:

$$\begin{aligned} I(F)(x) &= (F \circledast \phi)(x) && (4)\\ &= \sum_{i \in \mathbb{Z}^d} F[i]\phi(x-i) && (5) \end{aligned}$$

The continuous domain convolution can be approximated[2] as follows:

$$\begin{aligned} f * w &\approx I(F) * w && (6)\\ &= \left(\sum_{i \in \mathbb{Z}^d} F[i]\phi(x-i) \right) * w \qquad ; \text{using } (5) && (7)\\ &= \sum_{i \in \mathbb{Z}^d} F[i](\phi(x-i) * w) && (8)\\ &= \sum_{i \in \mathbb{Z}^d} F[i]w_\phi(x-i) && (9) \end{aligned}$$

[2]Theoretically, assuming no aliasing is introduced in the sampling step (i.e., the signal is bandlimited), f can be recovered exactly using the sinc as the interpolation kernel. In practice, the implementation of the sinc interpolation is unrealizable and aliasing may be present, hence the approximation nature of the claim.

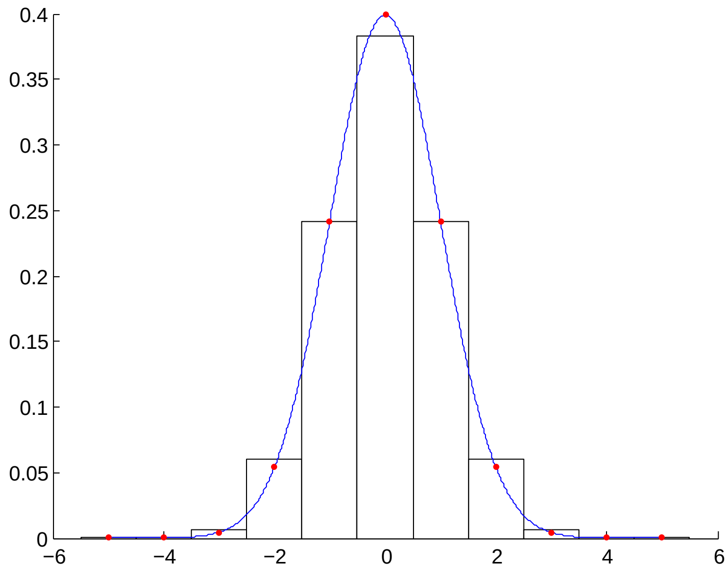

Figure 2: A one-dimensional Gaussian with standard deviation of 1. The dots denote the uniform samples of the Gaussian. The heights of the bars denote the coefficients of the interpolation-based approximation kernel using a nearest neighbour interpolant. Adapted from (Hummel & Lowe, 1989).

Applying the sampling operator, $S(\cdot)$, to both sides of (9), yields:

$$
\begin{aligned}
S(f * w) &\approx S\left(\sum_{i \in \mathbb{Z}^d} F[i](w_\phi(x-i))\right) && (10)\\
&= \sum_{i \in \mathbb{Z}^d} F[i] S(w_\phi(x-i)) && (11)\\
&= \sum_{i \in \mathbb{Z}^d} F[i] W_\phi[x-i] && (12)\\
&= F \circledast W_\phi && (13)
\end{aligned}
$$

□

For additional details and empirical comparisons, see (Hummel & Lowe, 1989; van den Boomgaard, 2001).

References

van den Boomgaard, R. (2001). Gaussian convolutions: numerical approximations based on interpolation. In *Scale-Space* (pp. 205–214).

Hummel, R. & Lowe, D. (1989). Computational considerations in convolution and feature-extraction in images. In J. Simon (Ed.), *From Pixels to Features* (pp. 91–102). Elsevier Science Publishers.

Koenderink, J. (1984). The structure of images. *Biological Cybernetics*, *50*, 363–370.