Curvature of Isophotes in an Image

Konstantinos G. Derpanis
York University
Department of Computer Science and Engineering
kosta@cs.yorku.ca

Version 1.0

October 18, 2005

This note summarizes the measurement of curvature along isophotes in a luminance image as presented in [1].

For a luminance image $L(x, y)$ where x, y are the Cartesian coordinates in the image plane, the equation of the boundary of the isophote (referred to as a blob by the authors [1]) is defined as $L(x, y) = L_0$ (see Fig. 1). The curvature of this boundary is found by implicitly differentiating the definition of the isophote twice with respect to the x coordinate. This yields two equation, specifically the first and second derivatives of y with respect to x which can be combined to explicitly solve for d^2y/dx^2, yielding,

$$\frac{d^2y}{dx^2} = \frac{-L_y^2 L_{xx} + 2L_x L_y L_{xy} - L_x^2 L_{yy}}{L_y^3} \tag{1}$$

The desired curvature κ is defined in the special case where the x-axis is tangent to the isophote. This is done by reexpressing the local structure in terms of a local coordinate system defined by the local structure of the image itself, specifically the *first-order gauge coordinates*[1]. In this case $L_x = 0$ and the solution reduces to,

$$\kappa = -\frac{L_{xx}}{L_y}. \tag{2}$$

Next the derivation of Eq. (1) is summarized. To reiterate, the defining equation of the isophote is given as,

$$L(x, y) = L_0. \tag{3}$$

[1]The *first-order gauge coordinates* are defined by the gradient direction and its perpendicular direction.

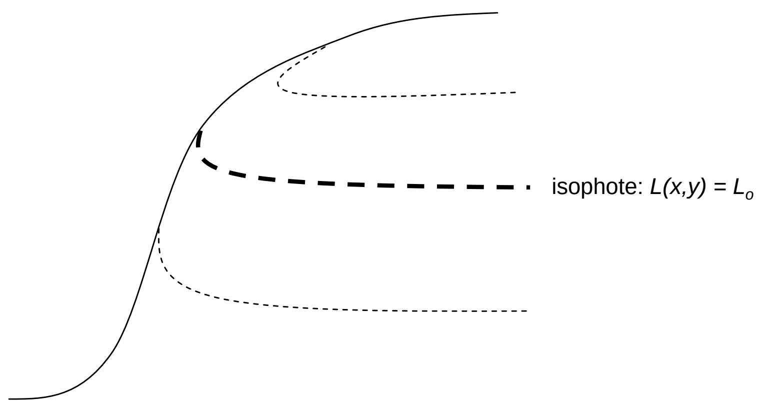

Figure 1: Image isophote.

The first derivative of y with respect to x is arrived at by implicit differentiation of the isophote (3),

$$\frac{dy}{dx} = -\frac{L_x}{L_y} = -L_x L_y^{-1}. \tag{4}$$

Differentiating (4) once again with respect to x, yields,

$$\frac{d^2y}{dx^2} = -L_y^{-1} L_{xx} - L_y^{-1} L_{xy} \frac{dy}{dx} + L_x L_y^{-2} L_{yx} + L_x L_y^{-2} L_{yy} \frac{dy}{dx}. \tag{5}$$

Substituting (4) into (5), yields,

$$\frac{d^2y}{dx^2} = -L_y^{-1} L_{xx} + L_y^{-1} L_{xy} L_x L_y^{-1} + L_x L_y^{-2} L_{yx} + L_x L_y^{-2} L_{yy} L_x L_y^{-1} \tag{6}$$

$$= -L_y^{-1} L_{xx} + L_y^{-2} L_{xy} L_x + L_x L_y^{-2} L_{yx} - L_x L_y^{-3} L_{yy} L_x \tag{7}$$

$$= -\frac{L_y^2 L_x x}{L_y^3} + 2 L_x L_{xy} L_y^{-2} - \frac{L_x^2 L_{yy}}{L_y^3} \tag{8}$$

$$= \frac{-L_y^2 L_{xx} + 2 L_x L_y L_{xy} - L_x^2 L_{yy}}{L_y^3} \tag{9}$$

References

[1] J.J. Koenderink and W. Richards. Two-dimensional curvature operators. *Optical Society of America - A*, 1988.