Quadrature filters

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Band-pass quadrature filters represent popular means in computer vision for estimating local multi-scale image information, such as the energy, phase, (radial) frequency and orientation/angular frequency. This note reviews the mathematical properties of quadrature filters in continuous space. In practice, care must be taken when realizing their discrete counterparts to avoid aliasing (i.e., violating the *Nyquist theorem* by undersampling). We begin with the one-dimensional quadrature filter and conclude with a discussion of two-dimensional generalizations.

Central to the definition of quadrature filters is the *Hilbert transform*. The Hilbert transform of a signal, $f(x)$, is defined as:

$$\begin{aligned} f_h(x) &= (h * f)(x) & (1)\\ &= \int_{-\infty}^{\infty} f(\tau)h(x-\tau)d\tau, & (2)\end{aligned}$$

where $*$ symbolizes the convolution operator and $h(x)$ is the point spread function given by,

$$h(x) = -\frac{1}{\pi x}, \tag{3}$$

with transfer function

$$H(\omega) = i \cdot \mathrm{signum}(\omega), \tag{4}$$

where i denotes the complex unit and

$$\mathrm{signum}(\omega) = \begin{cases} 1, & \omega > 0 \\ 0, & \omega = 0 \\ -1, & \omega < 0 \end{cases}. \tag{5}$$

Thus, the amplitude information of the Hilbert transformed signal remains the same while a phase shift of $\pm\pi/2$ is introduced for positive and negative frequencies, respectively.

A real-valued signal and its Hilbert transform can be combined into a complex valued signal, termed the *analytic signal*, $f_a(x)$, by the following transform,

$$f_a(x) = f(x) - i f_h(x). \tag{6}$$

The point spread function used to realize the analytic signal is given by,

$$a(x) = 1 + \frac{i}{\pi x} \tag{7}$$

with transfer function (see Appendix A for derivation),

$$A(\omega) = \begin{cases} 2, & \omega > 0 \\ 1, & \omega = 0 \\ 0, & \omega < 0 \end{cases}. \tag{8}$$

In other words, the analytic signal is a signal that has no negative frequency components.

(Band-pass) *quadrature filters* as traditionally used in the computer vision literature transform a real-valued signal, $f(x)$, to an analytic signal, $f_a(x)$, with the addition of weighting the frequency components. The quadrature filter transform is defined in the frequency domain as,

$$F_q(\omega) = \begin{cases} 2F(\omega), & \omega > 0 \\ 0, & \text{otherwise} \end{cases}. \tag{9}$$

Examples of quadrature filters in the literature include: Gabor filters[1,2] (Gabor, 1946; Daugman, 1980; Daugman, 1985), log-Gabor/Normal filters (Granlund & Knutsson, 1995) and Gaussian derivatives (Freeman & Adelson, 1991). For a comparative study of quadrature filters, the reader is referred to (Boukerroui, Noble & Brady, 2004).

In practice, quadrature filters can be used to estimate the local amplitude or energy[3], $A(x)$, and local phase, $\phi(x)$, as follows,

$$A(x) = \sqrt{\mathrm{Re}[f_q]^2 + \mathrm{Im}[f_q]^2} \tag{10}$$

$$\phi(x) = \arctan\frac{\mathrm{Re}[f_q]}{\mathrm{Im}[f_q]}, \tag{11}$$

where $\mathrm{Re}[\cdot]$ and $\mathrm{Im}[\cdot]$ correspond to the real and complex portion of the signal, respectively.

A popular generalization of quadrature filters to 2D (and more generally to nD) is the directional formulation (Jähne, 2005). In this case, quadrature filters are defined with respect to half spaces which are chosen with respect to a certain direction, formally,

$$F_q(\omega) = \begin{cases} 2F(\omega_x, \omega_y), & \omega \cdot \hat{\mathbf{n}} > 0 \\ 0, & \text{otherwise} \end{cases}, \tag{12}$$

where $\omega = (\omega_x, \omega_y)$ denote the spatial frequency coordinates, $\hat{\mathbf{n}}$ is a 2D unit vector defining the half-space and $\cdot$ symbolizes the inner product operator. Hence, in multiple dimensions the transfer function of the analytic signal preserves the frequency components on one side of the half-space. A drawback of the directional formulation is that it is not isotropic. To address the anisotropy of the half-space formulation, several authors (e.g., (Freeman & Adelson, 1991)) have proposed combining the quadrature filter results from multiple orientations.

More recently, Felsberg and Sommer (Felsberg & Sommer, 2001) proposed a vector-valued, isotropic, two-dimensional generalization of the analytic signal, they termed the *monogenic signal*. The proposed transform is founded on the *Riesz transform* rather than the Hilbert transform. However, for 1D signals the Riesz transform is equivalent to the Hilbert transform. For further details of the *monogenic signal* the reader is pointed to (Felsberg & Sommer, 2001; Jähne, 2005).

[1] Gabor filters are not strictly in quadrature since they have non-zero negative frequencies and a non-zero DC component. To address the DC related issue several authors have proposed correction schemes (e.g., (Heeger, 1988)).

[2] A common justification for the use of Gabor filters is their "optimality". It should be stressed that this optimality is attained with respect to a specific metric as defined in the *uncertainty principle*. This metric measures the joint spread/localization of a function in the spatial and spectral domains. However, there are an infinite number of plausible alternative metrics that could be substituted which yield different classes of optimal filters (Stork & Wilson, 1990).

[3] There is quite a bit of inconsistency in the literature in the usage of the term *energy*. Some authors define the squared amplitude as energy, while others use amplitude and energy interchangeably.

References

Boukerroui, D., Noble, J. & Brady, M. (2004). On the choice of band-pass quadrature filters. *Journal of Mathematical Imaging and Vision*, *21(1)*, 53–80.

Daugman, J. (1980). Two-dimensinal spectral analysis of cortical receptive field profile. *Vision Research*, *20(10)*, 847–856.

Daugman, J. (1985). Uncertainty relation for resolution in space, spatial frequency, and orientation optimized by 2D visual cortical filters. In *Journal of the Optical Society of America-A*.

Felsberg, M. & Sommer, G. (2001). The monogenic signal. *IEEE Transactions on Signal Processing*, *49(12)*, 3136–3144.

Freeman, W. & Adelson, E. (1991). The design and use of steerable filters. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, *13(9)*, 891–906.

Gabor, D. (1946). Theory of communication. *JIEE*, *93(3)*, 429–459.

Granlund, G. & Knutsson, H. (1995). *Signal Processing for Computer Vision*. Kluwer.

Heeger, D. (1988). Optical flow from spatiotemporal filters. *International Journal of Computer Vision*, *1(4)*, 279–302).

Jähne, B. (2005). *Digital Image Processing, sixth edition*. Springer-Verlag.

Stork, D. & Wilson, H. (1990). Do Gabor functions provide appropriate descriptions of visual cortical receptive fields? *Journal of the Optical Society of America-A*, *7(8)*, 1362–1373.

A Transfer function of an analytic signal

In this section, the derivation of the transfer function of an *analytic signal*, $F_a(\omega)$, is provided. Let $f_a(x)$ denote the impulse response of an analytic signal. Taking its Fourier transform (denoted $\mathcal{F}\{\cdot\}$), yields,

$$
\begin{aligned}
F_a(\omega) &= \mathcal{F}\{f_a(x)\} && (13)\\
&= \mathcal{F}\{f(x) - i f_h(x)\} && (14)\\
&= \mathcal{F}\{f(x)\} - i\mathcal{F}\{f_h(x)\} && (15)\\
&= F(\omega) - i\Big(i \cdot \mathrm{signum}(\omega) F(\omega)\Big) && (16)\\
&= F(\omega)\Big(1 - i^2 \cdot \mathrm{signum}(\omega)\Big) && (17)\\
&= F(\omega)\Big(1 + \mathrm{signum}(\omega)\Big) && (18)\\
&= \begin{cases} 2F(\omega), & \omega > 0 \\ F(\omega), & \omega = 0 \\ 0, & \omega < 0 \end{cases} . && (19)
\end{aligned}
$$