Signal Rescalings in Linear Scale-Space

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In this note we review the consequences in linear scale-space of rescaling a signal; the contents of this note are primarily adapted from the presentation in (Lindeberg, 1998). The results presented in this note play a central role in the formulation of scale invariant interest point detectors (e.g., (Mikolajczyk & Schmid, 2004)).

Briefly, the key idea in linear scale-space theory is the systematic removal of details from an image in order to describe the image structure in a fine-to-coarse manner. Alternatively, scale-space theory can be thought of as the theory of selecting a measurement aperture (weighted average over a support proportional to its scale parameter) (ter Haar Romeny, 2003). Given any continuous signal $I : \mathbb{R}^N \rightarrow \mathbb{R}$ (i.e., initial image), its linear scale-space representation $L : \mathbb{R}^N \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is defined as the solution to the (heat) diffusion equation, given by the following partial differential equation,

$$\frac{\partial L}{\partial t} = \frac{1}{2}\nabla^2 L \tag{1}$$

$$= \frac{1}{2}\sum_{d=1}^{D} \frac{\partial^2}{\partial x_d \partial x_d} L, \tag{2}$$

with initial condition,

$$L(\mathbf{x}; 0) = I(\mathbf{x}). \tag{3}$$

The solution to (2) can be expressed as the convolution of the initial image with the Gaussian kernel $G : \mathbb{R}^N \times \mathbb{R}^+ \rightarrow \mathbb{R}$,

$$G(\mathbf{x}; t) = \frac{1}{(2\pi t)^{N/2}} e^{-\sum_{i=1}^{N} x^2/(2t)}, \tag{4}$$

where $t = \sigma^2$ represents the standard deviation of the Gaussian. The scale-space representation given by (2) is realized by the specification of several axioms that formalize the notion of “uncommitment”, such as, linearity, shift invariance and various ways of formalizing the idea that new structures are not created from fine-to-coarse, for example, non-creation of new level curves (Koenderink, 1984); for further details on the axiomatic development of scale-spaces see (Koenderink, 1984; Alvarez, Guichard, Lions & Morel, 1993; Lindeberg, 1993) and for textbook treatments see (Lindeberg, 1993; ter Haar Romeny, 2003).

In the sequel, we consider the relationships between a signal and its rescaled copy.

Let two signals denoted I and I' be related as follows,

$$I(x) = I'(x'), \tag{5}$$

where,

$$x' = sx. \tag{6}$$

The scale-space representations of I and I' are denoted $L(\cdot;t)$ and $L'(\cdot;t')$, respectively, in their respective domains.

Next, we establish the relationship between L and L' (Lindeberg, 1998):

$$\begin{aligned}
L(x;t) &= G(x) * I(x) \qquad \text{; * denotes convolution} && (7)\\
&= \int_{-\infty}^{\infty} G(x - x'')I(x'')dx'' && (8)\\
&= \int G(x - x'')I'(sx'')dx'' \qquad \text{; using signal relation (5)} && (9)\\
&= \frac{1}{s}\int G(x - \frac{x'''}{s})I'(x''')dx''' \qquad \text{; change of variables, } x''' = sx'' && (10)\\
&= \frac{1}{s}\int \left(\frac{1}{\sqrt{2\pi t}} e^{-\frac{(x - x'''/s)^2}{2t}}\right) I'(x''')dx''' && (11)\\
&= \int \left(\frac{1}{\sqrt{2\pi s^2 t}} e^{-\frac{(sx - x''')^2}{2s^2 t}}\right) I'(x''')dx''' && (12)\\
&= \int \left(\frac{1}{\sqrt{2\pi t'}} e^{-\frac{(x' - x''')^2}{2t'}}\right) I'(x''')dx''' \qquad \text{; using coordinate relation (6)} && (13)\\
&= L'(x', t'), && (14)
\end{aligned}$$

where,

$$t' = s^2 t. \tag{15}$$

The mth order spatial derivatives satisfy (Lindeberg, 1998):

$$\frac{\partial^m L(x;t)}{\partial x^m} = \frac{\partial^m L'(x';t')}{\partial x^m} \tag{16}$$

$$= \frac{\partial^m L'(x';t')}{\partial x'^m}\left(\frac{\partial x'}{\partial x}\right)^m \tag{17}$$

$$= \frac{\partial^m L'(x';t')}{\partial x'^m} s^m \qquad ;\ x' = xs \Rightarrow \partial x'/\partial x = s \tag{18}$$

Note that in (18) the derivatives are **not** scale invariant. To yield scale invariant derivative measures, the following γ-*normalized operators* are introduced (Lindeberg, 1998),

$$\frac{\partial}{\partial \xi} = t^{\gamma/2}\frac{\partial}{\partial x}, \tag{19}$$

which corresponds to the change of variables,

$$\xi = \frac{x}{t^{\gamma/2}}. \tag{20}$$

In terms of the γ-normalized derivative operator (19) we have the following relationship (Lindeberg, 1998):

$$\frac{\partial^m L(x;t)}{\partial \xi^m} = (t^{\gamma/2})^m \frac{\partial^m L(x;t)}{\partial x^m} \tag{21}$$

$$= (t^{\gamma/2})^m s^m \frac{\partial^m L'(x';t')}{\partial x'^m} \qquad ;\ \text{using (18)} \tag{22}$$

$$= (t^{\gamma/2})^m s^m (t'^{\gamma/2})^{-m} \frac{\partial^m L'(x';t')}{\partial \xi'^m} \qquad ;\ \text{using (19)} \tag{23}$$

$$= (t^{\gamma/2})^m s^m (t'^{-m\gamma/2}) \frac{\partial^m L'(x';t')}{\partial \xi'^m} \tag{24}$$

$$= (t^{\gamma/2})^m s^m ((s^2 t)^{-m\gamma/2}) \frac{\partial^m L'(x';t')}{\partial \xi'^m} \qquad ;\ \text{using (15)} \tag{25}$$

$$= (t^{\gamma/2})^m s^m (s^{-m\gamma} t^{-m\gamma/2}) \frac{\partial^m L'(x';t')}{\partial \xi'^m} \tag{26}$$

$$= s^m (s^{-m\gamma}) \frac{\partial^m L'(x';t')}{\partial \xi'^m} \tag{27}$$

$$= s^{m(1-\gamma)} \frac{\partial^m L'(x';t')}{\partial \xi'^m}. \tag{28}$$

Given that the spatial position and scale parameters satisfy (6) and (15), respectively, setting $\gamma = 1$ in (28) leads to the scale invariant derivative relationship,

$$\frac{\partial^m L(x;t)}{\partial \xi^m} = \frac{\partial^m L'(x';t')}{\partial \xi'^m}, \tag{29}$$

or equivalently,

$$t^{m/2}\frac{\partial^m L(x;t)}{\partial x^m} = t'^{m/2}\frac{\partial^m L'(x';t')}{\partial x'^m}. \tag{30}$$

Generally, the amplitude of spatial derivatives in the scale-space representation decrease with scale (Lindeberg, 1993). The introduction of the normalized derivative operator counteracts the decreasing trend.

References

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