Singular Value Decomposition: Review

Konstantinos G. Derpanis

August 15, 2005.

In this note the *Singular Value Decomposition* (SVD) is reviewed. The SVD representes one of the most basic and fundamental concepts of linear algebra. The SVD was introduced for real square matrices by Beltrami and Jordan in the 1870's, for complex matrices by Autonne in 1902 and for general matrices (i.e., real/complex and square/non-square) by Eckhart and Young in 1939. For a survey on the early history of the SVD, the interested reader is pointed to (Stewart, 1993). Also, for further details on the SVD and its applications, see (Golub & Van Loan, 1996; Strang, 1988). In the sequel, this note concentrates on the development of the SVD for the real matrix case of arbitrary dimensions.

Theorem: Let $\mathbf{A} \in \mathbb{R}_r^{m\times n}$ where $m \times n$ denote the matrix dimensions and r the matrix rank. There exist orthogonal matrices $\mathbf{U} \in \mathbb{R}^{m\times m}$ and $\mathbf{V} \in \mathbb{R}^{n\times n}$ such that

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top \tag{1}$$

where

$$\Sigma = \begin{pmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{2}$$

$\mathbf{S} = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$ and $\mathbf{0}$ denotes the zero matrix of appropriate dimensions.

Proof: Since the matrix $\mathbf{A}^\top\mathbf{A}$ is non-negative definite the eigenvalues of $\mathbf{A}$ denoted $\lambda_i, i = 1, \ldots, n$ are all ≥ 0 (Strang, 1988). Next, rearrange the eigenvalues such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0 = \lambda_{r+1} = \cdots = \lambda_n$.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be the corresponding sorted set of orthonormal eigenvectors,

$$\mathbf{V}_1 = \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_r \end{pmatrix} \tag{3}$$

$$\mathbf{V}_2 = \begin{pmatrix} \mathbf{v}_{r+1} & \cdots & \mathbf{v}_n \end{pmatrix}. \tag{4}$$

and $\mathbf{S} = \mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_r}) \equiv \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$.

We have,

$$\mathbf{A}^\top\mathbf{A}\mathbf{V}_1 = \mathbf{V}_1\mathbf{S}^2, \tag{5}$$

and with further rearranging yields,

$$\mathbf{S}^{-1}\mathbf{V}_1^\top\mathbf{A}^\top\mathbf{A}\mathbf{V}_1\mathbf{S}^{-1} = \mathbf{I}, \tag{6}$$

where $\mathbf{I}$ denotes the identity matrix.

Also,

$$\mathbf{A}^\top\mathbf{A}\mathbf{V}_2 = \mathbf{V}_2\mathbf{0}, \tag{7}$$

and with further rearranging yields,

$$\mathbf{V}_2^\top \mathbf{A}^\top \mathbf{A}\mathbf{V}_2 = \mathbf{0}. \tag{8}$$

Let $\mathbf{U}_1 = \mathbf{A}\mathbf{V}_1\mathbf{S}^{-1}$. Notice that from (6) that $\mathbf{U}_1$ is an orthogonal matrix (i.e., $\mathbf{U}_1^\top \mathbf{U}_1 = \mathbf{I}$). Select $\mathbf{U}_2$ such that $\mathbf{U} = \begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{pmatrix}$ is an orthogonal matrix.

Finally, putting everything together:

$$\begin{aligned}
\mathbf{U}^\top A\mathbf{V} &= \begin{pmatrix} \mathbf{U}_1^\top \mathbf{A}\mathbf{V}_1 & \mathbf{U}_1^\top \mathbf{A}\mathbf{V}_2 \\ \mathbf{U}_2^\top \mathbf{A}\mathbf{V}_1 & \mathbf{U}_2^\top \mathbf{A}\mathbf{V}_2 \end{pmatrix} && (9) \\
&= \begin{pmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{U}_2^\top \mathbf{U}_1 \mathbf{S} & \mathbf{0} \end{pmatrix} && (10) \\
&= \begin{pmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} && (11) \\
&= \mathbf{\Sigma}. && (12)
\end{aligned}$$

Thus, $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$ **Q.E.D.**.

The r diagonal elements of $\mathbf{S}$ coupled with the $n - r$ zero elements that form the diagonal of $\mathbf{\Sigma}$ are termed the *singular values* of $\mathbf{A}$. The singular values correspond to the square roots of the eigenvalues of $\mathbf{A}^\top \mathbf{A}$. The columns of $\mathbf{U}$ and $\mathbf{V}$ are termed the *left* and *right singular vectors* of $\mathbf{A}$, respectively. The left and right singular vectors correspond to the orthonormal eigenvectors of $\mathbf{A}\mathbf{A}^\top$ and $\mathbf{A}^\top \mathbf{A}$, respectively.

References

Golub, G. & Van Loan, C. (1996). *Matrix Computations* (Third Ed.). John Hopings Press.

Stewart, G. (1993). On the early history of the singular value decomposition. *SIAM Review*, *35(4)*, 551–566.

Strang, G. (1988). *Linear algebra and its applications* (Third Ed.). Harcourt Brace Jovanovich College Publishers.