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THE APPLE GEDANKEN EXPERIMENT

1. The Quantum World

When we look at elementary particles such as electrons, we witness behaviour that is markedly different from what we see around us in the macroscopic world. For example, when an electron is presented with two possible paths, it can take both! Such phenomena manifest themselves in what we will call the *quantum world* in which the typical length scale is very small (nm = 10^{-9} m) and the typical temperature is very low (mK range). At higher length or temperature scales, the weird phenomena get blurred, and the quantum world reduces to ours.

The Apple gedanken experiment is intended to expose these phenomena in a simple setting, and to elicit the formalism that will be used to describe them mathematically.

2. The Experiment

The setups involve one or two rooms connected by corridors. Each room has two entry doors labeled "1" and "2", and two exit doors labeled "y" and "n". The idea is to bring people drawn randomly from a large population and have them enter our setup one by one. We don't let the next person in until the current one comes out. The person exits from the "y" door if it likes to eat apples and from the "n" door otherwise.

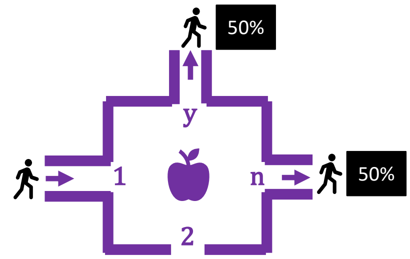
Our job is to observe and record from which door the person exits. After processing a large number of persons, we can produce "*y/n exit stats",* i.e. the percentage of people exiting from the "y" door and the percentage of those exiting from "n".

It should be noted that we cannot see inside rooms or corridors; we only see each person before entry and after exit.

Note that in most setups, we use only one of the entry doors of the room, and in that case, you can ignore the other entry door.

2.A) *One Room, One Entry Door, Two Exit Doors*

In this basis setup, we let people enter from Door 1 and record from which door they exit. We observe that:



A.1 After entering, the person always eventually comes out from one of the two exit doors.

A.2 After observing a large number of persons, we find that **50%** exit from "y" and **50%** from "n".

(Same findings if we let the people in from Door 2.)

Observation **A.1** (person-in, person-out) may seem obvious, but it is actually significant. It says that persons are *conserved*, i.e. no person disappears or is created inside, and no person splits into two and comes out from both doors. This finding persists in all setups, and it explains why the exit stats always add up to 100%. As we shall see, this finding underlies two key elements of the quantum formalism: *Normalization* and *Unitarity*. The former describes the subject (here, the person) and dictates that it must be conserved (the sum of the probabilities of all outcomes is 1). The latter refers to the device through which the subject passes (here, the room) and dictates that it must preserve the normalization.

Observation **A.2** (50-50 stats) can be explained if we make some assumption about the people or about the population from which they were drawn. Three possibilities exist:

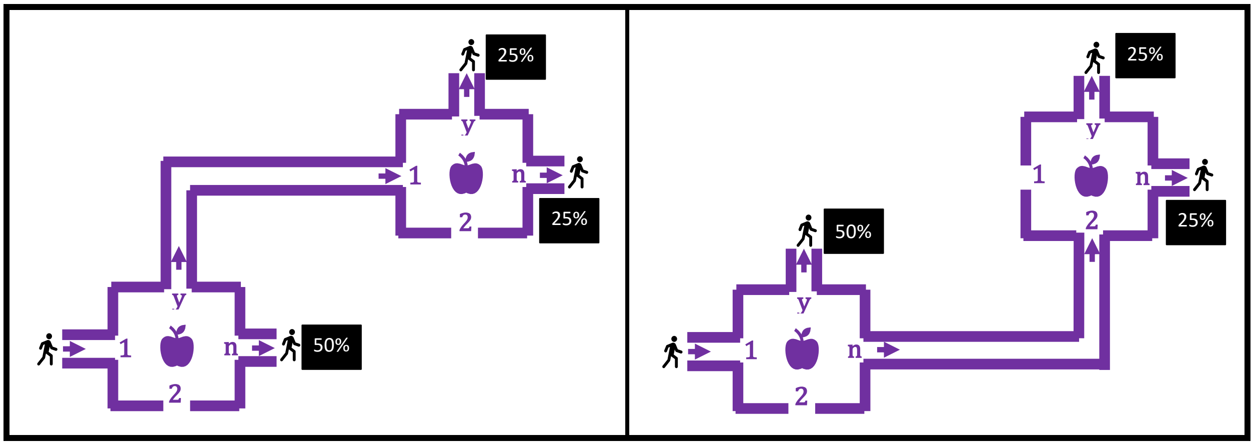
- **The Predisposition Hypothesis**
  Each person possesses an inherent inclination to respond one way or another to the "apple liking" question, and *half* the people in our population like apples; *half* don't.
- **The Random-Choice Hypothesis**
  Each person chooses the exit door by tossing a *fair* coin. Hence, liking or disliking apples is a random choice made when the question was posed.
- **The Mixed Hypothesis**
  This is a blend of the above two: some of the people in the population follow the first hypothesis while the rest follow the second.

As we look at other setups, we will test these assumptions to see if any can explain all findings.

- Argue that the predisposition hypothesis is what we normally adopt in our macroscopic world when we look at things or make measurements.
- Show that the mixed hypothesis produces the observed stats for *any* mixing ratio.

2.B)   Two Rooms, One Entry Door, Three Exit Doors

We add a second room and a corridor that connects one of the exit doors of the lower room to an entry door of the upper room. As shown, this can be done in two ways, but the findings are the same in both. We discuss below the setup depicted on the left. We observe that:



B.1 After entering, the person always eventually comes out from one of the three exit doors.

B.2 The exit stats: **50%** from the exit door of the lower room, and **25%** from each exit door of the upper.

(Same findings if we let the person in from Door 2.)

Observation **B.1** confirms our earlier conclusion regarding normalization.

Observation **B.2** presents a challenge to two of our hypotheses. The "predisposition hypothesis" cannot be right as it would predict that no one should exit from the "n" door of the upper room. The "mixed hypothesis" also fails because it cannot lead to such stats for any mixing ratio. The "random-choice" predicts the observed stats exactly.

Since our successful random-choice hypothesis is based on probabilities, let us review two rules that allow us to combine probabilities:

1) If two events are independent (e.g. obtaining two heads after tossing a coin twice) then the probability of both occurring is the *product* of their individual probabilities.
2) If an event can occur in two mutually exclusive ways (e.g. obtaining different faces after tossing a coin twice) then its probability is the *sum* of their individual probabilities.

As an example of applying the first rule, let us compute the probability of exiting from the "y" door of the upper room: for this event to occur, the person must choose the "y" door in the lower room and also choose the "y" door in the upper. And since these two choices are independent, we multiply their individual probabilities. The answer is thus ½ x ½ = ¼ or 25%.

- Show that the predisposition hypothesis leads to 50% exiting from the "n" door of the lower room and 50% from the "y" door of the upper.
- For the mixed hypothesis with a mixing ratio of 50-50, i.e. half the people are pre-committed (i.e. have a predisposition one way or the other) and half choose randomly, show that 37.5% will exit from the "y" door of the upper room, thereby falsifying this assumption.
- Show that the mixed hypothesis fails to produce the observed stats for all mixing rations, not just 50-50.
- Show that the random-choice hypothesis produces the observed stats.
- Apply the two probability combining rules to compute the probability of obtaining a sum of 11 after rolling a pair of dice.
  Hint: this sum can occur in two mutually exclusive ways (Rule #1) each of which involves two independent events (Rule #2).

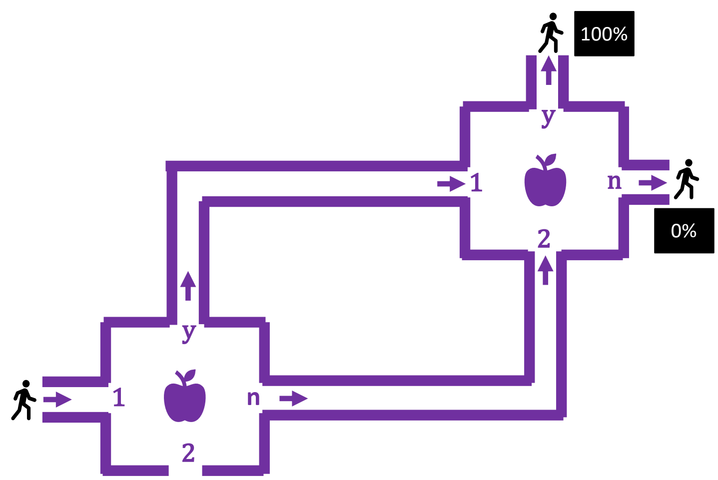
2.C) Two Rooms, One Entry Door, Two Exit Doors

We add a second corridor to connect the other exit door of the lower room to the other entry door of the upper.

We observe that:

C.1 After entering, the person always eventually comes out from one of the two upper exit doors.

C.2 After observing a large number of persons, we find that **100%** of the people exit from the "y" door of the upper room, and **none** from its "n" door.



(Opposite finding if we let the person in from Door 2, i.e. 100% from "n", none from "y".)

Observation **C.1** confirms our earlier conclusion regarding normalization.

Observation **C.2** is troublesome because our random-choice hypothesis predicts a y/n exit stat of 50-50, rather than 100-0. We can see this by noting that for every 100 persons entering the lower room, 50 will choose "y" and enter the upper room from its Door 1 where 25 of them will exit from "y" and 25 from "n". The same argument applies to the other 50, thus leading to an exit stat of 50-50. You can also reach the same result by applying our two rules: existing from "y" can occur in two ways: by entering the upper room from its door 1 (which has a probability of ½) and then choosing to exit from "y" (probability = ½), or from its door 2. Combining these two leads to: ½ x ½ + ½ x ½ = ¼ + ¼ = ½. And similarly for the "n" door.

The random-choice hypothesis may thus need to be adapted or, failing that, totally abandoned.

One way to adapt it follows this train of thought: we found that for every 100 persons entering the lower room, 25 will reach the "n" of the upper room via its Door 1 and 25 via its Door 2. If we subtract these two 25s instead of adding them, we can achieve the observed 0%. But what does that mean? Let us try to look for meaning in math first. In order to turn addition into subtraction, we can make one of the two numbers, say the one associated with Door 2, negative! This ploy may indeed save the day provided we can deal with three repercussions, two of which are technical, and one is logical:

1) Probabilities cannot be negative! If we want to see a negative contribution from Door 2 then that contribution has to be called something other than probability. We will call it ***amplitude*** (a number between -1 and +1) and use it as in: "the amplitude of exiting from n via Door 2". And to maintain our traditional interpretation of probability as a positive number between 0 and 1, we derive it from the amplitude by ***squaring*** it. From now on, we will only work with amplitudes, computing and combining them using our two rules after replacing the word "probability" with "amplitude", and in the end, we compute probabilities (stats) by squaring the final amplitudes. For example, the random-choice hypothesis would now dictate that the amplitude of a choice is $1/\sqrt{2}$. Working with amplitudes and converting to probabilities at the end leads to different answers than working with probabilities. For example, two mutually exclusive events will have their amplitudes a and b added, which leads to a final probability of $(a+b)^2$ which is different from $a^2 + b^2$ (what we get if we add the probabilities).
2) Will the amplitude ploy lead to the observed stats? Exiting from "y" occurs in two ways: by entering from Door 1 (which has an amplitude of $1/\sqrt{2} \times 1/\sqrt{2} = ½$) or by entering from Door 2 (which has an amplitude of also $1/\sqrt{2} \times 1/\sqrt{2} = ½$). Adding these two yields ½ + ½ = 1, so the probability is 1^2 or 100%. Repeating this for exiting from "n" and associating an amplitude of $-1/\sqrt{2}$ for entering from Door 2 leads to ½ - ½ = 0, so the probability is 0^2 or 0%.
3) This leaves a serious logical problem to do with interpreting the cancelation that took place at the "n" door. If many persons can be in the setup at the same time, then one can imagine, for the upper room, that those coming from 1 to the "n" door, and those coming from 2 to the "n" door, would meet and somehow agree to abort their exit plans and exit instead from the "y" door. But our experiment allows only one person at a time to be in the setup, so whom did that person meet to make it change its plan? It looks like we have to accept that when the person was asked to choose between two exit doors in the lower room, it chose both, i.e. it actually walked in both paths, and when it met itself in the upper room, "they" agreed to exit from the "y" door. The technical word for this is ***superposition***: the person will be in a superposition of two states, one chooses "y", and one chooses "n".

We will formalize these observations in the next section.

3. Matrix Mechanics

Recall that our two rules, which we will henceforth refer to as the "*amplitude arithmetic rules*", involve addition (for mutually exclusive alternatives) and multiplication (for independent ones). Hence when we an event that has two alternatives each of which involves independent events, we end up with an expression involving the sum of two products. Such expressions conjure up expressions we get when a 2x2 matrix multiplies a 2x1 column vector:

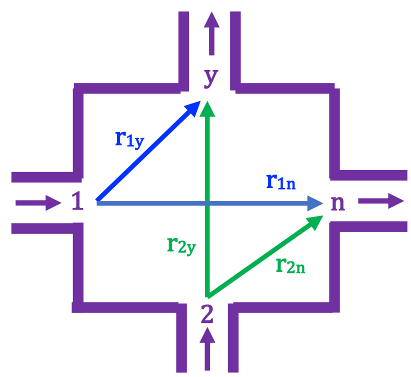
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

This suggests that using matrices and vectors might be a good way to describe the behaviour of quantum systems. Furthermore, our observation regarding superposition also suggests that the mathematical structure to be used must be able to treat two entities as one (one person makes two choices at the same time), and column vectors fits this requirement too.

Based on the above, we formalize our analysis of the Apple Gedanken experiment as follows: We represent the state of each subject (person) in the experiment by a column vector; and re-present each device (room) by a matrix. The action of a device on a subject is captured by the product of the matrix and the vector. The product leads to a new vector which we interpret as the state of the subject after exiting the device, e.g. the person after exiting from the room.

Each room in our experiment will be represented by a 2x2 matrix whose rows are labeled by its entry doors (1 and 2) and whose columns are labeled by its exit doors (y and n).

$$\text{Room} = \begin{pmatrix} r_{1y} & r_{1n} \\ r_{2y} & r_{2n} \end{pmatrix}$$



The four matrix elements in it are the amplitudes of the four possible choices, e.g. r_{1n} is the amplitude of entering the room from Door 1 and exiting from Door n.

For our gedanken experiment, we will use the following matrix for both rooms:

$$R_1 = R_2 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

A person entering from Door 1 (2) will be represented by the column vector P_1 (P_2) whose rows are labeled by entry door numbers and whose entries are the door entry amplitudes:

$$P_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, P_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Note that each vector is normalized: $1^2 + 0^2 = 1$.

Let us use this formalism to follow a person along its journey in Setup C:

We know that our person will enter from Door 1 of the lower room. This means it starts in the state P_1. After exiting that room, its state can be computed from:

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

The state shows equal amplitudes (and thus equal probabilities) for exiting from either door, which formalizes what we said earlier about exiting from both doors. Note that the output state is still normalized: $(1/\sqrt{2})^2 + (1/\sqrt{2})^2 = 1$. Note how the room matrix transformed the state vector from one labeled by entry doors (1 and 2) to one labeled by exit doors (y and n).

The person is now in the two corridors. The top corridor turns its Room 1, Door "y" exit component "y" to Room 2, Door 1 entry component. Similarly, the lower corridor turns its "n" exit to a "1" entry. Hence, the entry state vector to Room2 is the same as above.

Next, the person goes through the second room. Upon exit from it, its state is computed from:

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \times \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The exit vector is normalized, and it shows that the probability of exiting from "y" is 1^2 = 100% and from "no" is 0^2 = 0%, as expected.

That the state vector started off being normalized is by construction: we made sure of that. But is it an accident that it stayed normalized after getting multiplied (twice) by a matrix? No, not an accident thanks to the unitarity of the room matrix. *Unitary matrices preserve the length of the vectors they multiply*.

We noticed that the labeling of the two components of the state vector is based on entry doors before entering and on exit doors after exiting. This makes sense given how the rows / columns of the room's transforming matrix are labeled. But to avoid any ambiguity, we sometime represent the state vector not as a column matrix but as *a vector in a two-dimensional plane*, as this would make the labels explicit. Expressed like this, the above states of the person become:

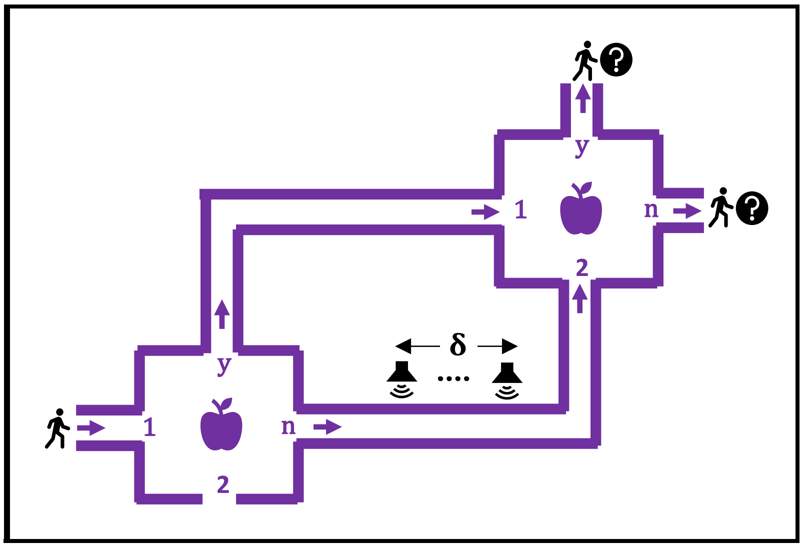
- Before entering Room 1: 1**|1>** + 0**|2>**
- After exiting Room 1: 1/√2**|y>** + 1/√2**|n>**
- Before entering Room 2: 1/√2**|1>** + 1/√2**|2>**
- After exiting Room 2: 1**|y>** + 0**|n>**

where **|1>**, **|2>** and **|y>**, **|n>** denote unit vectors along the plane's axes.

- If |a> and |b> are unit vectors, is the state: (2|a> + 3|b>)/√13 normalized?
- Normalize the state 3|a> + 4|b> by multiplying it by a number.
- Multiply the room matrix ([[1,1], [1,-1]])/√2 by the unnormalized state: 2|a> + 4|b>. Show that the room's unitary matrix preserved the length of the state vector.
- Show that the matrix ([[1,1], [1,1]])/√2 is not unitary.
- *Hint: pick any vector and show that the matrix changes its length.*

4. Complex Amplitudes

Start with Setup C and install speakers (**δ** of them) in the lower corridor and stream through them messages and ads that promote the health benefits of eating apples. After observing a large number of persons, we find that the exit stats vary with **δ** as shown in the table to the right of the diagram (the entry for **δ** = 0 means no speakers installed, i.e. same as Setup C).



δ	y %	n %
0	100	0
1	77	23
2	29	71
3	1	99
4	17	83
5	64	36
6	98	2
7	88	12
8	43	57

To apply the formalism, we need a matrix to represent the speakers. Its entries must only affect the lower part of the superposition (where the speakers were installed) so it must look like this:

$$\begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix}$$

Where X is some number that depends on **δ**. Applying it to the state after Room 1 yields:

$$\begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} \times \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ X/\sqrt{2} \end{pmatrix}$$

Next, we apply the matrix of Room 2 to get the final output state:

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \times \begin{pmatrix} 1/\sqrt{2} \\ X/\sqrt{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+X \\ 1-X \end{pmatrix}$$

This final state cannot be right because its normalization (the sum of the squares of its entries) is $(1+X^2)/2$, which is 1 only if X=0, but we want X to depend on **δ**. In fact it should be an oscillatory function of n **δ** if we are to reproduce the observed stats above. One way to make X oscillatory and keep the state normalized is to allow **complex amplitudes**. We will adopt this from now on and define the probability as the **square of the modulus of the amplitude**.

Specifically, we will set X to a pure ***phase*** (a complex number with modulus 1 and argument **δ**):

$$X = e^{i\delta} \quad \text{(recall that } e^{i\delta} = cos\delta + isin\delta\text{)}$$

In that case, the square of the two amplitudes become:

$$|\frac{1}{2}(1+e^{i\delta}|^2 = \frac{1}{4}\{\,[1+cos(\delta)\,]^2 \;+\; sin^2(\delta)\,\} = \;\cos^2(\frac{\delta}{2})$$

$$|\frac{1}{2}(1-e^{i\delta}|^2 = \frac{1}{4}\{\,[1-cos(\delta)\,]^2 \;+\; sin^2(\delta)\,\} = \;\sin^2(\frac{\delta}{2})$$

As you can see, the state is normalized (for all values of **δ**) and the "y"/"n" probabilities are periodic in **δ**/2. The reproduce the observed stats exactly.

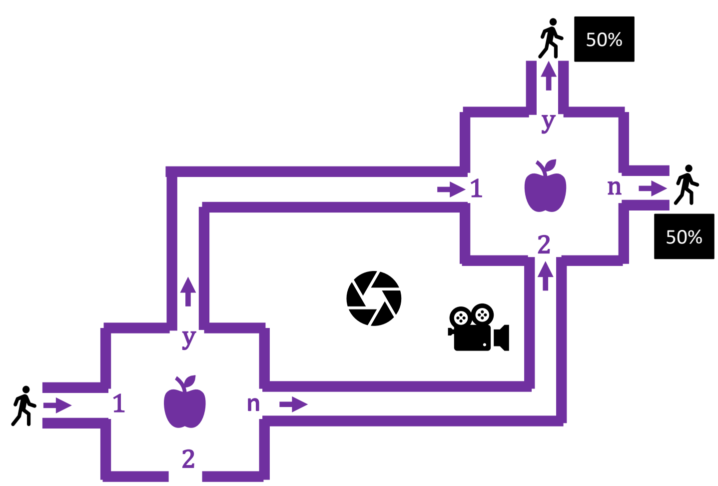
5. Measurement & Collapse

The matrix-based framework is easy to use and does produce observed results. Nevertheless, it does not really "explain" what is happening when the system is in a superposition state. Did the person actually take both paths in our experiment? It seems that it did, but did it achieve that by splitting into two or by cloning itself? To find out, let us install cameras or video recorders in the corridors of Setup-C and take a peek.

We observe that:

- We always see the person in one of the corridors, *never in both*.
- The exit stats become *50-50*.

Hence, the *superposition is gone*—it has **collapsed**. In fact, any attempt you come up with to determine which path the person took will lead to a collapse. Such attempts are **measurements**.



We have seen that the rooms do transform the state of the system, and so do the presence of speakers, but they don't lead to a collapse. Why is installing a camera so much different? The answer stems from **information**. As long as information (about the system) does not leak to some other system, everything works well and stays in the quantum world, but once a leak is present, the system returns to our classical world. Speakers can affect the person, but they don't tell you anything about its location. Cameras do. That is why any attempt to *"actually understand"* the quantum behaviour is futile because the act of measuring will force the system to leave the quantum world.

The idea that measurement can affect what is being measured is true not only in the quantum world but also in our world. In ours, however, its effect can be controlled. When you measure the length of a table using a measuring tape, pressing the tape against the table could bend it a little and change its length, but this minuscule change is negligible, and you can make it as small as you want. But if you try to measure the distance between two electrons by looking at them, i.e. by illuminating them and looking at the reflected light, the energy in the light's photons will knock them out place, thereby completely changing the sought distance.

The rooms do not force the person to choose one door or another; it can choose both. But the act of looking after the person exits, forces the person to be either present at a door or not. This is because our measuring device (our eye) can only see a full person or no person. The same applies when the experiment is conducted with elementary particles: the exit detectors can either detect a particle or fail to detect, and hence, they force the superposition to collapse.

Remarks

- I came up with this experiment idea to break the conceptual barrier in the learning curve of anything quantum. Using familiar objects and situations, foundational quantum concepts are exposed, and some aspects of the formalism is introduced. But even though the experiment is imaginary, *every element in it can actually be realized in the lab!* In fact, its setups were inspired by an experiment that was performed in the 19th century by two physicists, Mach and Zehnder, to show how light waves interfere. Their apparatus, the Mach-Zehnder Interferometer MZI, uses light waves (which as we know now are composed of photons) instead of our persons; beam splitters (which give photons a choice to reflect or transmit through) instead of our rooms; and glass plates (which change the phase) instead of our speakers. In their time, light sources emit zillions of photons per second, which means zillions of persons would be in the rooms at the same time. This completely blurs out the quantum effects and allows for an explanation based on how the collective behaves. Today, we can run this very MZI by sending one photon at a time with electronics ensuring the next photon cannot enter before the current one is detected.
- Waves are *not* one photon at a time so you can invoke arguments like "they split in the first room"; "they met in the second room"; "they interfered constructively or destructively"; and so on. No such simple interpretation works for one photon at a time. Superposition is indeed a genuinely quantum effect.
- A 60W light bulb consumes energy at the rate of 60 Joules each second. Each photon carries energy given by *hf,* where h=6.6×10^{-34} is Plank's constant, and f is the light frequency (about 6×10^{14} for mid-range visible light). If all the energy is emitted as light, the number of photons emitted per sec = $60 / (6.6 \times 10^{-34} \times 6 \times 10^{14}) \approx 2 \times 10^{20}$. A far cry from one-photon-at-a-time!
- Superposition allows two things (two choices, two locations, …) to happen at once. Two for the price of one, or parallelism, has always been the holy grail of computing. We can see it in our gedanken if the problem we seek to solve were: *"Is it raining in either corridor?"*. For Setup-C, the answer is easy; send one person and see if its clothes are wet upon exit. Only one person, one trip is needed. On a classical computer, you would need either two persons or one person going twice, once per corridor. This is a factor of 2 speedup! As we shall see, the Deutsch-Jozsa algorithms leverages this to show that a quantum computer can solve certain problems in 1 shot compared to an exponential number of shots for a deterministic classical computer.
- It sounds negative (for QC) that superposition collapses if we take a peek, but we can turn this to our advantage: if we communicate via superposition, no intruder can possibly eaves-dropping on our conversation! Any attempt to listen in will collapse the state and expose the intruder. We will see later that quantum cryptography leverages this to ensure security levels that are guaranteed by the laws of physics.